AN

ELEMENTARY TREATISE

ON

ANALYTICAL GEOMETRY

EMBRACING

PLANE CO-ORDINATE GEOMETRY

AND AN

Introduction to Geometry of Three Dimensions.

DESIGNED AS A TEXT-BOOK FOR COLLEGES AND SCIENTIFIC SCHOOLS.

BY

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PREFACE.

CO-ORDINATE GEOMETRY is the basis of the modern or Analytical method in Mathematical Science. Not only does it afford, in its applications, the best possible exercises in Algebraic reasoning; but, by familiarizing the mind with the ideas of variables and functions, and graphically illustrating them, it is the suitable preparation for the study of the Differential Calculus and the higher branches of Algebra. I have endeavored to adapt this treatise to the use, both of those who wish to study the Conic Sections by the co-ordinate method, and of those who intend to pursue the higher branches of Analysis.

The common rectangular equations of the Ellipse, the Parabola and the Hyperbola have been derived from the usual and familiar definitions of these curves; and demonstrations of their properties, founded exclusively upon these equations, occupy the first parts of Chapters V., VI. and VII. In this portion of the work, extensive application has been made of the "eccentric angle" in the ellipse, and, in the case of the hyperbola, of a similar auxiliary angle, of which the use is suggested in Salmon's Treatise on Conic Sections. In the Chapter on the Hyperbola, the most useful properties and equations of the three conic sections have been generalized, with especial reference to the manner in which these curves present themselves in Astronomy.

The latter parts of the same Chapters are devoted to more general equations of the curves, and to their discussion by means of the method of combined equations, which is the basis of the "abridged notation." In Chapter VIII., the general equation of the second degree is treated by the method of transformation of co-ordinates. It has been my object, in this Chapter, to give methods of deriving from an equation in the general form all the circumstances which relate to the position and form of the curve; and also to discuss all the special forms of the equation, not examined in the previous Chapters. If it be desired to abridge the course by leaving out the topics referred to in this paragraph, the student may omit the following portions, which are not necessary to the perusal of the remainder of the work: Chapter III.; Chapter IV. from Art. 123 to the end; Chapter V. from Art 151; Chapter VI. from Art. 202; Chapter VII. from Art. 254; and the whole of Chapter VIII.

In Chapter IX., I have attempted to classify the principles employed in finding the equations of Geometrical Loci, and to explain and illustrate them fully by examples solved in the text, and a carefully graduated series of examples for practice. Chapter X. contains a full explanation of the notation employed in Solid Geometry, its application to the plane and the straight line, and a brief notice of the various surfaces of the second degree. It is believed that this Chapter will be found an adequate preparation, in this branch, for the study of standard authors in Analytical Mechanics.

The knowledge of Geometry and Trigonometry essentially prerequisite to the commencement of this study is very slight. I have therefore avoided demonstrations founded upon any geometrical principles or trigonometric formulæ except those fundamental ones mentioned in Art. 2.

Nothing tends to impress mathematical principles and methods upon the mind so thoroughly as the solution of numerical examples, with their *verifications*. Such solutions and verifications will be found in the text of those Articles in which practical methods are explained. Unsolved examples both of a numerical and of an algebraic character are added at the ends of the Articles or sections. To the more difficult examples, especially to those in which algebraic demonstrations are required, are appended hints for their solution.

April, 1869.

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ANALYTICAL GEOMETRY.

CHAPTER I.

APPLICATION OF ANALYSIS TO PLANE GEOMETRY.

ART. 1. Quantity is that which can be measured by a unit of its own kind; as length, time, weight. A magnitude of any kind is represented by the number of units it contains. Algebra, the science of quantity in general, treats of number as the representative of magnitude, and its processes are applicable to all kinds of quantity.

Analysis is the term applied in mathematics to that treatment of a subject, in which appropriate magnitudes, represented by symbols, are introduced into algebraic equations and subjected to algebraic processes.

It is the object of this treatise to explain a system by which Plane Geometry is subjected to this treatment.

2. This system is based upon a few fundamental principles of Geometry; namely, the doctrines of parallel lines and of similar triangles, and the property of a right-angled triangle, that the square of the hypothenuse is equal to the sum of the squares of the sides.

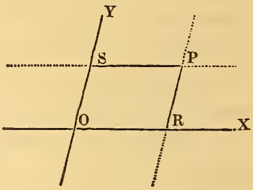
The method of investigation presupposes a knowledge of the principles of Algebra, so far as necessary to the solution and discussion of equations of the first and second degrees; of the usual notation of Trigonometry for angles and their functions, *sin*, *cos*, etc.; and of the fundamental relations,

$\frac{sin}{cos} = tan$, $sin^2 + cos^2 = 1$, etc., which are consequences of the definitions of the functions and of the property of a right triangle above referred to.

Position.

3. The magnitudes which, in the system to be explained, enter into algebraic equations are those which determine the position of a point in a plane.

Let OX, OY be two fixed intersecting lines of the plane, and P be any point. From P let PR and PS be drawn, each parallel to one of the fixed lines and terminated by the other; then the distances PR and PS expressed in terms of some unit of length determine the position of P.



The fixed lines of reference, OX and OY, are called the *axes*, and the lengths PS and PR are called the *co-ordinates* of the point P. The former is distinguished as the *abscissa* and the latter as the *ordinate*, and the axes to which they are parallel are called respectively the *axis of abscissas* and the *axis of ordinates.* It is customary in the figures to draw the former horizontal or parallel to the bottom of the page, so that OX is the axis of abscissas and OY, of ordinates.

A point is said to be *given*, when the number of units or parts of a unit contained in its abscissa and ordinate are given; these numbers are themselves called the co-ordinates of the point. Thus, a point is given, when we say its abscissa is $3\frac{1}{2}$, and its ordinate is 2.

4. Since SPRO is a parallelogram, OR = SP, and therefore OR may be taken as the abscissa of P. For the same reason, it is the abscissa of every point on the line PR, produced indefinitely either way; that is to say, every point of a straight line parallel to OY, the axis of ordinates, has the same abscissa. In like manner, every point of the line SP parallel to the axis of abscissas has the same ordinate, and OS might be regarded as

their common ordinate; but it is usual, of the distances PR and PS, to draw only the former to represent the ordinate, letting OR, the part of the axis of abscissas cut off, represent the abscissa. The ordinate of every point in the line OX is evidently zero, and its abscissa is its distance from the point O. The ordinate of every point in the line OY is also its distance from O, while its abscissa is zero. The point O where the axes intersect is called the *origin;* each of its co-ordinates is zero.

5. The symbols used to denote the co-ordinates of a point are those usually devoted in algebra to unknown quantities—namely, x and y, because, in the class of problems we shall first meet, they will be the unknown quantities. By common consent, x has been appropriated to the abscissa, and y to the ordinate. For this reason the axis of abscissas is marked in the figures by a capital X, and is frequently called the *axis of* X; and the axis of ordinates is distinguished by a capital Y, and called the *axis of* Y. The abscissas of known points will usually be denoted by such symbols as x', x'', x_1, x_2, etc., the ordinates by y', y'', y_1, y_2, etc., and the points to which these co-ordinates refer, correspondingly, by P', P'', P_1, P_2, etc.

Systems of Co-ordinates.

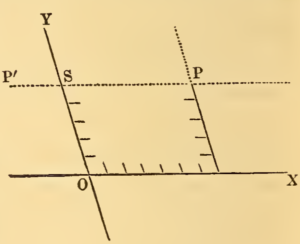
6. It is evident that one of the co-ordinates alone will not define completely the position of a point; and indeed, owing to the dimensions of space, it is impossible that one numerical element should determine position in a plane. To define position upon a *line*, we would need to assume a fixed point of reference on the line, or origin of distances, as well as a unit of distance, and then the position of a point would be defined by the number of units in its distance from the origin; just as to define an instant of time, we assume a fixed instant or epoch of reference, and a unit of duration. But, if the point is not confined to a certain line, but only to a surface in which it is free to move in all directions, *two* determining elements are required; and if it is not limited at all, but free to take any position in space, *three* are necessary. These determining elements, of whatever kind they may be, have received the general name of *co-ordinates.* In treating of the relative positions of bodies

in space, or questions of "Solid Geometry," the three dimensions of space render it necessary to use three co-ordinates, but in Plane Geometry we need two, and two only, because all the points treated are confined to a single surface. As an example of the same principle, we may observe that the position of a point on the earth's surface is defined by two co-ordinates, its latitude and longitude; while, to define completely the position of any point relatively to the earth, we need a third co-ordinate, as height above the level of the sea, or distance from the earth's centre.

7. Different systems of co-ordinates for position in a plane may be proposed. The system we have just described, in which the point is referred to two fixed lines, is the principal one in use. It is called the system of *Cartesian* co-ordinates, being the invention of Des Cartes. Besides this, there is another method in use called "Polar co-ordinates," in which the position of a point is defined by its *distance* from a fixed point called the pole, and the *direction* in which this distance is measured—the co-ordinate determining the latter being an angle. Cartesian co-ordinates are called *rectangular* or *oblique*, according as the axes are at right angles or obliquely inclined. Rectangular co-ordinates is a special case of oblique, in which the axes may be inclined at any angle whatever. Formulæ are frequently very much simplified by their use, but care must be taken not to extend to the general case those which apply *only* to rectangular co-ordinates.

Construction of Points.

8. Suppose now, we are required to find the point whose abscissa is 7, and whose ordinate is 5. The axes being drawn and a unit of length assumed, we might lay off the abscissa on the axis of X, either on the right or on the left of the origin; it is customary to lay it off to the *right*, and to lay off the ordinate *upward*. Therefore, laying off seven of the assumed units on the axis of X, from the origin toward the right hand, and from the point so reached five units upward on a parallel to the axis of Y, we find



the required point. We should have arrived at the same point had we first laid off five units on the axis of Y, and then seven units on a parallel to the axis of X. The operation is called the *construction* of a given point, and the point is referred to as the point seven, five, and written thus (7, 5), the values being enclosed in brackets, and that of x written first. The construction of a point consists of two parts; in the process used above, we first constructed the value of the abscissa or x, by constructing the point (7, 0) and drawing the parallel, for every point of which $x = 7$; we then construct the value of y, by finding the point of this line, for which $y = 5$. Or we might construct the line for every point of which $y = 5$, and then find the point of this line for which $x = 7$; that is, the point common to the two lines.

9. From the rule to lay off the value of the abscissa to the *right*, it evidently follows that adding to the value of x carries the point to the right, and *subtracting* from it, to the *left*. Adding a number of units to the value of x (leaving that of y unchanged) is therefore equivalent to moving the point P so many units to the right along the dotted line PP′. We can thus *increase* the value of x, or conceive it increased to any extent. Subtracting from the value of x, would move P on the same line an equivalent number of units to the left, which motion would *decrease* the value of x, until, having subtracted the whole value of x, we arrive at the point S, whose abscissa is 0. But as motion to the left can take place indefinitely, as well as motion to the right, we can, in this sense, subtract from the value of x a greater number of units than it contains. Thus, moving the point P, of the figure, 10 units to the left, we arrive at the point P′, whose abscissa is $7 - 10$. The subtraction indicated is arithmetically impossible, but in algebra the result is -3, which therefore properly expresses the value of the abscissa of P′, a point 3 units to the *left* of the axis of Y. Addition and subtraction, being thus represented by two opposite and mutually destructive motions, are always equally possible; subtraction no longer implies the existence of some quantity to subtract from, and the "negative quantities" of algebra become equally intelligible with the "positive."

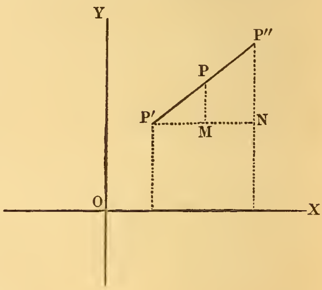
We may consider motion to the left as still *decreasing* the value of x, even when the point is already on the left of the axis of Y.

This sense of the word decrease will be distinguished from the ordinary one as *algebraic decrease;* so that, for a point on the left of the axis, motion to the left "algebraically decreases," though numerically increasing, the negative value of x.

In the same manner it may be seen that increase and decrease in the value of y correspond to motion upward and downward, and that a *negative* value of y properly expresses position *below* the axis of X. Thus we can construct a point with any given values whatever, positive or negative, laying off the positive values to the right and upward, the negative to the left and downward. It is, of course, arbitrary which directions we regard as positive, but whichever we select, the opposite must be considered negative. The capitals X and Y, in the figures, will always be placed on their respective axes in the positive directions from the origin.

Relative Position of Points.

10. As the position of a point depends upon the values of its co-ordinates, so the relative position of two points depends upon the differences of their co-ordinates. If, in passing from one position to another, the co-ordinates are both increased, the point moves upward and to the right; that is, it follows a direction intermediate to the positive directions of the axes. If they are both diminished, the direction is the reverse of this; if x is diminished and y increased, it passes upward to the left, etc. In general, let P$'$ and P$''$ be two given points, of which we wish to know the position of P$''$ relatively to P$'$. We have only to notice the signs of the differences $x'' - x'$ and $y'' - y'$, which are the values of P$'$N and NP$''$. In the figure they are both positive. Comparing P$'$ with P$''$, the differences $x' - x''$ and $y' - y''$ are both negative. These differences evidently determine both the relative direction of P$'$ and P$''$ and the distance between them.



11. *To find the co-ordinates of a point midway between two given points.* Let $x'\,y'$, $x''\,y''$, be the co-ordinates of the given

points, and let x and y be the co-ordinates of the point P, bisecting the line P′P″. Then P′M : P′N : : P′P : P′P″ : : 1 : 2, by similar triangles; that is, $P'M = \frac{1}{2} P'N$. But $x - x'$ represents the length of P′M, and $x'' - x'$, of P′N, therefore $x - x' = \frac{1}{2}(x'' - x')$. Similarly we prove $y - y' = \frac{1}{2}(y'' - y')$; hence

$$x = \tfrac{1}{2}(x' + x'') \qquad \text{and} \qquad y = \tfrac{1}{2}(y' + y'').$$

These results may be expressed thus: Each of the co-ordinates of the middle point is an arithmetical mean between the corresponding given co-ordinates. In deriving results, as above, from a figure and a geometrical property, the algebraic expressions put in the place of lines should all be positive.

12. The distance of given points, as P′ and P″, depends upon the lengths of P′N and NP″, and also upon the included angle P′NP″. If the axes are rectangular, the square of the distance P″P′ equals the sum of the squares of these lines, hence

$$P'P'' = \sqrt{(x'' - x')^2 + (y'' - y')^2}$$

is the formula for the distance of two points, when the axes are *rectangular*. As this formula is simpler than the corresponding one for oblique co-ordinates, which involves the angle between the axes,* questions in which the distance between points is concerned are usually treated by rectangular co-ordinates.

Examples.—Construct the points (— 1, 3) and (3, — 5),

Determine the co-ordinates of the point bisecting their distance.

Find the distance of each from the middle point, the axes being rectangular.

What are the co-ordinates of the point bisecting the distance of P′ from the origin, and the formula for that distance?

Ans. $x = \frac{x'}{2}, y = \frac{y'}{2}$; $OP' = \sqrt{x'^2 + y'^2}$.

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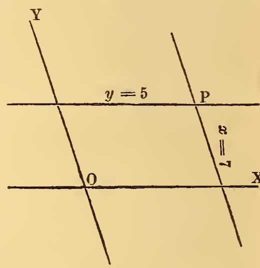
* If the axes be oblique, P′NP″ is the supplement of YOX. If then YOX (the angle between the positive directions of the axes) be represented by ω, trigonometry gives

$$P'P'' = \sqrt{(x'' - x')^2 + (y'' - y')^2 + 2(x'' - x')(y'' - y')\cos\omega},$$

which reduces to the above form, when $\omega = 90°$.

LOCUS OF AN EQUATION.

13. We have seen that two co-ordinates or determining elements must be known in order to fix the position of a point. These may be given directly, in algebraic language, by two equations, such as $x = 7, y = 5$; or indirectly by two equations between x and y, as $x + y = 12, x - y = 2$, from which the above values may be deduced. The equations $x = 7, y = 5$, taken together, are called the *equations of the point*. It was shown, in Art. 8, in which this point was constructed, that the equation $x = 7$, taken alone, enables us to construct a line parallel to the axis of Y, on which the point is to be found, because it contains all the points for which $x = 7$. This equation, therefore, or the single *condition* that its abscissa be 7, *limits* the point to a particular line, though leaving it yet undetermined. The single condition $y = 5$ also limits the point to a certain line, namely, a parallel to the axis of X; but the two conditions, taken together, determine the point as the one common to the two lines, because that is the only one which fulfils at the same time *both* conditions.

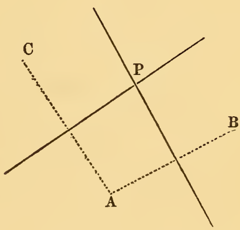


14. It will be seen hereafter, that each of the equations $x + y = 12$ and $x - y = 2$, is a condition which taken singly limits the point to a certain line, and that P, the point (7, 5), is the common point or intersection of these two lines. In general, every equation between x and y is a condition imposed upon the point to which it refers, which limits it to a certain line; this line is called the *locus* of the equation, that is, the *place* to which it restricts a point.

For example, take the simple equation $x = y$. This expresses the condition that the abscissa and ordinate of the point are equal. It evidently does not determine the point, because it is true of a variety of points as (1, 1), (2, 2), etc. Let a straight line be drawn through the origin bisecting the angle YOX; it will readily be seen that the abscissa and ordinate of every point of this line are equal, and that this is true of no other points; therefore the line is the locus of the equation $x = y$.

In like manner construct the locus of the equation $x = -y$, which expresses the condition that the abscissa and ordinate be equal with contrary signs.

15. Whenever, in Plane Geometry, a point is to be determined, there must be given *two* independent conditions. Each condition taken by itself limits the point to a certain line, which may be called its *locus.* When the *loci of the two conditions* are drawn, the solution is completed, for the common point or points of the two loci can alone fulfil both the conditions. Thus, if we wish to describe a circle through three given points of a plane, A, B and C, we may find its centre by the independent conditions, that it must be equally distant from A and B, and also that it be equally distant from A and C. A perpendicular bisecting AB is the locus of the first condition, because it can be proved that it contains *all* the points which fulfil that condition, and no others. A perpendicular bisecting AC is the locus of the other condition, and as the loci have but one common point P, there is one and but one solution. If A, B and C are in one straight line, the loci will be parallel, and there will be no solution.



The above may be called a *graphic* solution, because the loci of the conditions are *drawn.* In the analytic method, the conditions of a problem are thrown into the form of equations between the co-ordinates of the point sought, whose values are then found by the algebraic process of elimination.

Indeterminate Equations and Functions.

16. Since two conditions are thus necessary to determine a point in a plane, if but one is given, the problem is said to be *indeterminate.* So in algebra a single equation between two unknown quantities is called "an indeterminate equation," because insufficient to determine their values. Such an equation between x and y may in general be satisfied by a value of x, selected at pleasure, and a corresponding value of y determined from the equation; thus $x = 2y - 3$ is satisfied by $x = 0$ and $y = 1\frac{1}{2}$, $x = 1$ and $y = 2$, $x = 2$ and $y = 2\frac{1}{2}$, etc. Regarding, then, x as independent and

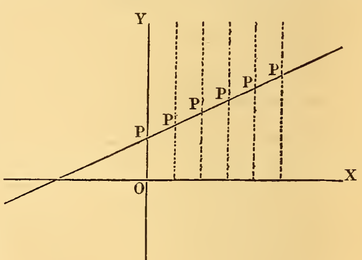
y as depending upon it, we see that the equation may be satisfied by an indefinite number of *sets of values* of x and y. A quantity depending upon another for its value, as y does here upon x, is said to be a *function* of it. The unknown quantities, x and y, of an indeterminate equation are called *variables;* because one of them, as x, may be regarded as capable of assuming any number of different values or varying *independently*, while the other, in order to satisfy the equation, must take certain corresponding values, and therefore varies *dependently* upon the former.

A function is said to be *explicit* when its value is directly *expressed* in terms of the independent variable, but *implicit* when the dependence is merely *implied* by the existence of an equation. For instance, if $x^2 + y^2 = 25$, y may either be regarded as an implicit function of x, or as the independent variable of which x is an implicit function. Solving for y, we have $y = \pm \sqrt{25 - x^2}$, in which y is an explicit function of x.

EXAMPLES.—Find sets of values satisfying $x + y = 12$; sets of values satisfying $x - y = 2$.

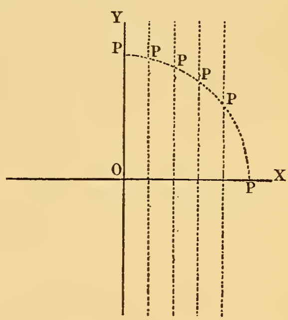
Make y an explicit function in each case.

17. Let x and y, in an indeterminate equation, stand for Cartesian co-ordinates; then, for each of the sets of values of x and y, a point may be constructed. Each of these points will be a point of the locus of the equation, and is said to *satisfy* the equation, because its co-ordinates satisfy the equation. The equation of the last article, $x = 2y - 3$ is satisfied by the points $(0, 1\frac{1}{2})$, $(1, 2)$, $(2, 2\frac{1}{2})$, etc. In order conveniently to find a number of points of the locus of an equation, make y an explicit function of x; thus $y = \frac{1}{2}(x + 3)$; then giving to x a number of equidistant consecutive values, as 0, 1, 2, 3, 4, 5, the corresponding values of y, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4 are readily found, and points P, P, P constructed as in Art. 8. It is evident in this case, from the value of y in terms of x, that any increase in the value of x produces an increase of half the amount in y. As a consequence of this, the points con-



structed lie on one straight line, as in the figure. This straight line is in fact the locus of the equation, but neither these points, nor any number of others which might be constructed with intermediate values of x, properly speaking, constitute the line. It may, however, be conceived as *described* by a *moving point* whose abscissa increases uniformly from the value zero, while its ordinate increases from the value $1\frac{1}{2}$, also uniformly, but at half the rate. If x pass through all possible values, positive and negative, the whole line will be described.

18. The law of the variation of a function is not generally so simple as in the above case. Take, for instance, the equation $x^2 + y^2 = 25$. Making y an explicit function, we have $y = \pm \sqrt{25 - x^2}$, in which if we give x the values 0, 1, 2, 3, 4, 5, we obtain for y the series of values, ± 5, $\pm \sqrt{24}$ (about 4.9), $\pm \sqrt{21}$ (about 4.6), ± 4, ± 3, 0. The positive values of y in this case *decrease* as the values of x *increase*, and the *rate* of decrease is *not uniform*, but becomes more and more rapid. If the abscissa of a point increase uniformly from zero, while its ordinate decreases from 5, being always the above function of its abscissa, it will describe a *curve*, because the rate of y is not uniform. If this curve were drawn, it would therefore *represent* the function, and its direction at any point would show the rate of decrease in y. For this reason, a number of points are frequently constructed for a function, and a line roughly traced through them to represent the function approximately; the axes in such a case are usually taken rectangular.



19. Constructing in this way the values of y, obtained above, for the function $y = \sqrt{25 - x^2}$, and tracing a line, we should hardly fail to notice that it approximates to the quadrant of a circle. Because the value of y *decreases* as that of x *increases*, it is called a *decreasing* function; the *rate* of decrease is variable, being evidently greater for a greater value of x. In the case of Art. 17, as y *increased with* x, the function was an *increasing* one. If the circle, with centre at the origin, and radius five units in length, be drawn,

we can prove that the co-ordinates of each of its points satisfies the equation $x^2 + y^2 = 25$; because they form the sides of a right triangle whose hypothenuse is the radius. All the quadrants of the circle are included, because the positive values of x give also negative values of y, and the corresponding negative values of x give the same positive and negative values of y. This circle, therefore, in the sense explained, *represents* the equation $x^2 + y^2 = 25$, which is therefore said to be the *equation of this circle.*

EXAMPLES.—Trace the lines representing the functions, $y = x - 2$, $y = x^2$, $y = 5 - 2x$.

Trace the locus of $x^2 - y^2 = 16$; of $x^2 + y^2 = 36$.

EQUATION OF A LINE OR LOCUS.

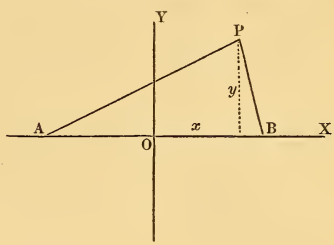
20. If a point move under a certain *condition*, or according to a certain *law*, it will describe a line. The equation, of which this line is the locus, expresses the condition or law, in the form of a *relation* between the co-ordinates of the point. The line will be called the *locus of the point* moving under the law, as well as the locus of the condition or of the equation which expresses it, and the equation will be called the *equation* of the line. Thus, $x = y$ is the equation of a straight line bisecting the angle between the axes, or of the locus of a point, so moving as to be always equidistant from the two axes; $x^2 + y^2 = 25$ is the equation of the locus of a point always 5 units distant from the origin. The equation of a line need not be written so as to make y an explicit function of x, for it connects x and y together, so that either may be considered a function of the other. If we wish to ascertain whether a given point is on the line of which we have the equation, we have merely to substitute the given values for x and y, and see if they satisfy the equation. Thus (7, 5) is on the line represented by $x + y = 12$, but (8, 3) is not.

21. As every *regular* or mathematical line or curve may be described by a point moving according to some *law*, every such line must have an equation. The law of the motion is equivalent to some *common property* of every point of the line, which may be made the definition of the line; as in the case of the circle above, the common property of whose points is their distance of 5 units from the origin. The equation $x^2 + y^2 = 25$ expresses a common

property of all the points, which is proved to be equivalent to this defining property. We have now to show how any law of motion, or common property of points, may be converted into an equivalent relation between co-ordinates; that is, into an equation between x and y.

22. Given the law of motion of a describing point, or a common property of all the points of a line, such as is involved in its definition, we proceed as follows: Construct a figure, (in which let P represent any point of the line, or the moving point in any of its positions,) by drawing the lines referred to in the conditions of the problem. Select lines as the axes of co-ordinates, and draw the co-ordinates of P, which mark by the letters x and y. These are called the *general co-ordinates* of the line, because P does not represent a particular point, but *any* point of the line. Then, by means of geometrical principles, establish a relation between x and y. As this relation must be true of every point of the line, it is the equation required.

For example, suppose P to move so that the sum of the squares of its distances, PA and PB, from two fixed points, A and B, shall be constantly 82, the distance between A and B being 8. Let the line AB be selected as the axis of X, and let the middle point of AB be the origin, and the axes rectangular. By the condition $PA^2 + PB^2 = 82$, and by right-angled triangles, since AO and OB each equal 4, $PA^2 = y^2 + (4 + x)^2$, $PB^2 = y^2 + (4 - x)^2$; hence adding, $2y^2 + (4 + x)^2 + (4 - x)^2 = 82$. Reducing, $x^2 + y^2 = 25$ is the equation required.



Example.—Find the equation, when the condition is that $PA^2 - PB^2 = 48$.

Algebraic Equations of Lines.

23. It is not necessary that the line of which the equation is sought should itself be drawn in the figure; on the contrary, its form may be entirely unknown, until revealed by the equation found,

as in the illustration given in the last article. For, $x^2 + y^2 = 25$ having been already found as the equation of a certain circle, we now know the form of the line described by P under the proposed condition.

When the form of the line is known, we take P anywhere on the line at random, and make use of the defining property of the line to establish the relation between x and y.

24. We shall, with the next Chapter, begin systematically to develop the equations of known lines in various positions, and to classify them by their equations, carrying this so far as to be able, on resuming, in Chapter IX., the consideration of "Geometrical Loci," to ascertain the position and form of the line represented by any equation of the first or second degree.

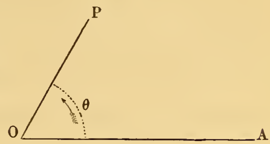
When the constant quantities which enter into the equation of a line are represented by letters to which any values may be assigned, the result is an *algebraic* or *general* equation, which includes a variety of *numerical* equations, and hence may represent any one of a certain class of lines. Thus, $x = 7$ was found to represent a parallel to the axis of Y, at a distance of seven units on its right; but $x = a$ may be made, by giving different values to a to represent any parallel to that axis; it is therefore said to represent *generally* a parallel to the axis of Y. Similarly, $y = b$ represents a parallel to the axis of X. Suppose now we had found an algebraic equation or *formula*, which, by giving different values to the constants, might be made to represent a straight line in any position whatever. Then every equation which can be reduced to this form will be known to belong to some straight line, and the general form, of the equation so reducible will be the *general equation* of the straight line. The algebraic equations also enable us to develop the properties of lines in a more general manner.

Polar Co-ordinates.

25. We shall now describe the other system alluded to in Art. 7, in which distance from a fixed point, and direction, are used to determine position.

Let O be a fixed point, and OA a straight line drawn from it toward the right, and let P be any point of the plane. O is called the *pole*, and OA is called the *initial line*, as it marks an *initial direction*, with which all other directions are compared.

Join OP, the position of P is evidently determined by the length of OP its distance from the pole, and the value of the angle POA. The distance OP is called the *radius vector*, and denoted by the symbol r, and the angle POA is denoted by the Greek letter θ; r and θ taken together are the *Polar co-ordinates* of P. The angular co-ordinate θ is reckoned from OA, over toward the left, as in the figures in trigonometry. The value of θ in the figure is less than $90°$; the value $\theta = 0°$ would indicate that the radius vector had the direction OA, or that P was on the line OA, and if θ be increased from that value, r remaining constantly the same, P will describe a circle with centre at the pole, completing it and returning to its primitive position, when $\theta = 360°$.



The direction of the point P being thus completely represented by the value of θ, its distance, r may be regarded as essentially positive. If $r = 0$, P is at the pole, whatever the value of θ, and by increasing r without limit, we may define any point of the plane by a positive value of r and a value of θ between $0°$ and $360°$. The method of constructing a point with given co-ordinates is simply to draw a straight line from O with the direction marked by θ, and then measure off on it from O a number of units equal to the value of r.

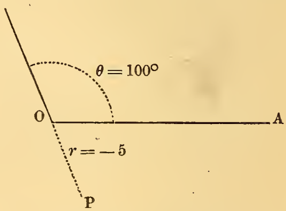
26. The values of two co-ordinates being necessary in this system to determine the position of a point, (except of the pole itself, which is determined by $r = 0$,) one given co-ordinate, or one equation between co-ordinates, restricts the point to a certain line or locus, as in the Cartesian system. Thus, $r = 5$ taken alone restricts the point to a circle with radius 5, and its centre at the pole; $\theta = 40°$ restricts the point to a straight line, having this inclination to the initial line, and terminated in one direction by the pole.

In an equation between r and θ, r should be regarded as a function of θ. Then by assuming values of θ, and determining from the equation the corresponding values of r, any number of points may be constructed which satisfy the equation. An *approximation* to the locus of the equation, may then be constructed by tracing a line through the constructed points; but it may be conceived as described by continuous motion, if a point P on a line uniformly revolving about the pole move in that line according to the law of the function; that is, so that its distance r from the pole shall always be the proper function of the angle θ, described by the line.

27. Equations between r and θ are of two kinds—those in which the angle θ occurs only through its trigonometric functions, as $r \cos \theta = 7$, and those in which the angle itself occurs, as $r = 3\theta$. In the first, r depends only upon the *direction* marked by θ. The value of θ may then be expressed in degrees, and no distinction is to be made between values which differ by 360°, as 10° and 370°, because they mark the same direction, and have the same functions. But in the other case θ is expressed in the arcual or circular measure, in which 2π, the length of the circumference of a circle whose radius is the unit, takes the place of 360°; and values which differ by 2π, though marking the same direction, will not give the same value of r. Thus, in $r = 3\theta$, the value of r uniformly increases from 0 to 6π, during the first revolution of the line; that is, while θ increases from 0 to 2π; and then, during the second revolution, continues to increase from 6π to 12π, while θ must be considered as passing from 2π to 4π. Thus the value of r may be increased indefinitely; the curve described is called the "Spiral of Archimedes."

Polar equations of the first class alone will be treated.

28. It may happen in a polar equation that some values of θ have, corresponding to them, *negative* values of r. Although negative values of r are not required, to define the position of certain points, as negative values of x and y were in the other system; yet such *corresponding values* of θ and r may be constructed. Thus, suppose $\theta = 100°$ gives for r the value -5, we draw a line from O, making this angle with OA, and then measure off 5 units on the line *produced through* O. This is consistent with the ordinary interpreta-



tion of negative values, and the consideration of such values may be necessary to the complete discussion of an equation.

Evidently the same point would be denoted by an equal positive value of r, and $\theta = 280°$. By admitting negative values of r, we might dispense with all values of θ above $180°$; then the whole line, drawn in the figure, would be represented by $\theta = 100°$, the part below the initial line containing points for which r must be taken as negative.

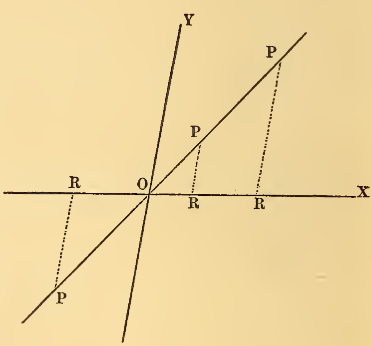
We may adopt a notation for a point given by polar co-ordinates similar to that of Art. 8, $(7, 100°)$, denoting the point whose radius vector is 7 units and its inclination $100°$; then $(-7, 280°)$ denotes the same point.

CHAPTER II.

THE STRAIGHT LINE.

29. In order to find the algebraic equation of the straight line, we must draw a representative line, unrestricted in its position with respect to the axes, and establish an equation between the co-ordinates of a point, taken at random on the line, and certain constants depending upon the *position of the line.* The equation must be true for *every* point of the line drawn, with the *same values* of the constants; that is, true for the different values of x and y corresponding to all points of the line, otherwise it does not represent the line. But *different* values being given to the constants, the equation must be capable of representing *any* straight line, otherwise it will not be what we are seeking—namely, a *general* formula for the straight line.

30. But, first let us consider a particular class of straight lines—namely, those which pass through the origin. Draw such a line, and the ordinates of several of its points. The triangles PRO thus formed are all similar, and therefore the ratio PR : RO or $y:x$ is constant for all points of the line. Let this ratio or the



quotient $\frac{y}{x}$ be represented by m; then we have for the equation

$$\frac{y}{x} = m, \text{or} \qquad y = mx.$$

The value of the constant ratio, m, determines the direction of the line, it is therefore called the *direction ratio.* In the figure, m is positive, for the co-ordinates x and y are either both positive or both negative, so that their quotient is always positive.

If the line had been drawn in a direction intermediate to the positive direction of one axis and the negative direction of the other, its points would have co-ordinates of different signs, and the value of m would be negative.

Whatever the value of m, the origin satisfies the equation $y = mx$; that is, the origin is always a point of the line. To construct the line with a given value of m, it is only necessary to construct one other point satisfying the equation; thus, if $m = 2$, that is, if we have the equation $y = 2x$, assume any value of x, as $x = 1$, then $y = 2$, and the point $(1, 2)$ is a point of the line. Joining this point with the origin we construct the line.

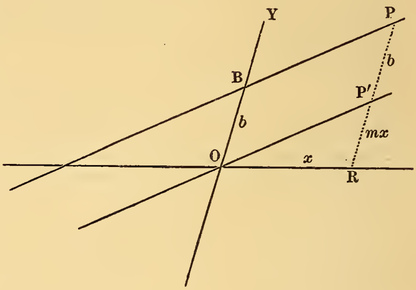
Hereafter we shall refer to a line given by its equation, as the line $y = 2x$, the line $x + y = 3$, etc.

EXAMPLES.—Construct $y = 5x$, $x + y = 0$, $x = 2y$.

Give the value of m in each case.

31. Evidently the smaller the value of m the more nearly does the direction of the line $y = mx$ approach to that of the axis of X. If we give m the value zero, the equation becomes $y = 0$, which represents that axis itself, for on the axis of X the ordinate is always zero.

Let $n = \frac{1}{m}$; then n is the value of the ratio $\frac{x}{y}$, and $x = ny$ represents the same line as $y = mx$. Now, suppose a line passing through the origin to revolve from right to left, and consider its equation both in the form $y = mx$ and $x = ny$. When the line coincides with the axis of X, $m = 0$. During the revolution the value of m increases and that of n decreases. As the line approaches the axis of Y, m becomes very great and n very small, and finally when it coincides with the axis of Y, $n = 0$, and m is said to be *infinitely great* or simply *infinite.* If the revolution be continued in the same direction from this position, n becomes negative and numerically increases, so that it still decreases algebraically; m also becomes negative and numerically decreases, so that it continues to increase algebraically until the line again coincides with the axis of X, when $m = 0$ and n is infinite. In general m and n have definite values, which are reciprocals of each other, and either form $y = mx$ or $x = ny$ may be used; but in the special cases in which one of these quantities is zero, the other is infinite, and but one form can be used, the resulting equations being $y = 0$ and $x = 0$, the equations of the axes.



32. Now let the line BP be drawn cutting the axis of Y in any point, B. Draw OP′ through the origin parallel to it, and let m represent the direction ratio of OP′. Drawing the ordinate of any point, P, we see that it consists of two parts, P′R the ordinate, corresponding to OR in the line OP′, and PP′, which, by the properties of the parallelogram, is equal in all cases to OB. Representing this part of the ordinate, which is constant, by b, we have

$$y = mx + b,$$

for the equation of the line. As BP has the same direction as OP′, m is its "direction ratio," and any change in the value of m would have the same effect on the direction of the line $y = mx + b$, as on $y = mx$. When m is positive, as in the figure, mx will be negative for negative values of x, so that y is less than b for points on the left of the axis of Y. For a certain negative value of x, mx will be numerically equal to b, and y will be zero, and for greater negative values of x, y will be negative. But when the value of m is negative, mx is negative for positive values of x, and y is less than b for points on the right of the axis. The *direction* of the line being determined by m, the value of b determines the *position* of the line, by determining the position of one of its points, B, and it is plain that by giving different values to b, we may produce the equation of any line parallel to $y = mx$. When $b = 0$ we have the line $y = mx$ itself, and when b is negative the line lies below OP′, and the part PP′ is subtracted from P′R. Thus,

$y = 2x + 2$, $y = 2x + 1$, $y = 2x$, $y = 2x - 1$, etc., is a series of parallel lines whose direction ratio is 2.

33. In $y = mx + b$, y is an explicit function of x, and the equation may be regarded as expressing that when $x = 0$, $y = b$, and that as x increases uniformly from the value zero, y increases uniformly from the value b, but at the comparative rate m. This is but a general statement of what was said in Article 17 of the equation $x = 2y - 3$, which was put in the form $y = \frac{1}{2}x + 1\frac{1}{2}$, in which $+\frac{1}{2}$ is the value of m, and expresses the comparative rate of the variation of y, and $1\frac{1}{2}$ is the value of y corresponding to $x = 0$.

Every equation of the first degree will take the form $y = mx + b$, if we make y an explicit function of x; and since every equation of this form represents some straight line, an equation between x and y of the first degree must represent a straight line. For this reason such equations have been called *linear* equations, and functions of the form $y = mx + b$, *linear* functions.

The equation of a straight line may also be written,

$$x = ny + a,$$

in which x is made an explicit function of y. This is the equation of a line parallel to the line $x = ny$ (which passes through the origin), and cutting the axis of X at the distance a from the origin to the right or left, according as a is positive or negative.

34. If the line is oblique to both the axes, we may express its equation in either of the forms $y = mx + b$, or $x = ny + a$, but if it is parallel to the axis of X, $m = 0$, and the first form gives for its equation

$$y = b.$$

If it is parallel to the axis of Y, $n = 0$, and the second form reduces to

$$x = a.$$

In these cases the co-ordinates are *not* variables, either of which is a function of the other, but one of them is constant in value, independently of the other.

35. The general equation of the first degree,

$$Ax + By + C = 0,$$

consisting of a term containing x, a term containing y, and a term independent of x and y, called the *absolute term*, represents some straight line, whatever the values of A, B and C. For if A, B and C have finite values, it may be reduced to either of the forms $y = mx + b$, or $x = ny + a$; if $A = 0$, that is, if the term containing x is wanting, to the form $y = b$; if $B = 0$, to the form $x = a$; if $C = 0$, to the form $y = mx$. This equation includes all classes of straight lines, and is, therefore, the *general equation of the straight line.*

EXAMPLES.—Reduce $2x - 3y + 5 = 0$ to the forms $y = mx + b$ and $x = ny + d$; $x + y = 3$ to both forms.

Reduce $3x = 2y$ to the form $y = mx$; $5x + 2 = 0$, to the form $x = a$.

CONSTRUCTION OF EQUATIONS.

36. Since an equation of the first degree always represents a straight line, to construct its locus it is only necessary to construct two of its points and to draw a straight line through them. Thus, any two of the points constructed in Art. 17 for the equation $x = 2y - 3$, with the knowledge that it represents a *straight* line, would serve to determine its locus.

The points usually constructed for this purpose are those in which the line cuts the two axes. The distances of these points from the origin, or the parts of the axes intercepted between the line and the origin, are called the *intercepts* on the axes. The intercept on the axis of X is the abscissa of that point of the curve which has zero for its ordinate; it is therefore found by letting $y = 0$, and deducing from the equation the corresponding value of x, and may be represented by the symbol x_0. In like manner the intercept on the axis of Y is found by letting $x = 0$, and deriving from the equation the corresponding value of y, and we shall denote it by the symbol y_0. Thus, in the equation $x + 3y = 3$, $y = 0$ gives $x_0 = 3$, and $x = 0$ gives $y_0 = 1$. If we measure off these distances respectively on the axis of X and Y, we shall have two points through which to draw a straight line, which being done we have *constructed* the equation $x + 3y = 3$.

If one of the terms belonging to the general equation is wanting, the line belongs to one of the particular classes of straight lines represented by $y = b$, $x = a$ or $y = mx$.

If to the first of these, b, the constant value of y is also the value of y_0, and the line is drawn parallel to the axis of X, so that there is no intercept on that axis. In the second case, $x_0 = a$, and there is no intercept on the axis of Y. In the third case, that is, when the absolute term is wanting, both intercepts equal zero, because the line passes through the origin, therefore some other point besides the origin must be found, thus in constructing $3x = 2y$, we find the point (2, 3) satisfies the equation, and therefore draw a line through this point and the origin.

EXAMPLES.—Find the intercepts and construct the lines, $x + y = 3$, $2y = 3 - 2x$, $4y = 3x$, $5x = 7$.

Give in each case the direction ratio, or value of m when reduced to the form $y = mx + b$.

37. General values of the intercepts may be found by using the general equation $Ax + By + C = 0$; thus,

$$x_0 = -\frac{C}{A} \text{ and } y_0 = -\frac{C}{B}.$$

When $C = 0$, both these values becomes zero, as above remarked. When $A = 0$, so that the equation may take the form $y = b$, x_0 takes the form $-\frac{C}{0}$, a finite quantity divided by zero, or infinity. x_0 is generally found by putting $y = 0$ in the equation; if, however, this is done in an equation of the form $y = b$, for instance, $y = 5$, we have the impossible result $0 = 5$. This impossible result indicates that there is *no value* of the intercept x_0, or that the line *does not* cut the axis of X. This must be distinguished from the result $x_0 = 0$, which indicates that the line *does* cut the axis of X at the origin. In the general solution this impossibility of the intercept is indicated by the form infinity taken by its value.

There is another peculiar form which the value of x_0 may take—namely, $\frac{0}{0}$, when both $A = 0$ and $C = 0$. This is called the *indeterminate* form, and indicates that x_0 may take *any value whatever;* that is, the line meets the axis of X in *every* point, or *coincides* with it. For if A and C are each zero, the equation reduces to $By = 0$ or $y = 0$, which is the equation of the axis of X itself.

Similar remarks apply to the peculiar forms which may be taken by the value of y_0.

Intersection of Loci.

38. When the common point or points of two lines are required, their equations being given, we have to find values of x and y, which satisfy the two equations at the same time.

A problem which gives rise to two equations, between x and y, is said to be *determinate*, in distinction from the "indeterminate" problems mentioned in Art. 16, because two equations are sufficient to determine the point sought; and the two equations when considered in connection are called *simultaneous equations;* because, arising from the same problem, they are to be satisfied *at the same time.* Thus, if the two conditions of a problem give the equations, $x = 2y - 3$, and $x + y = 3$, we find by elimination that they are satisfied *simultaneously*, by the values, $x = 1$, $y = 2$. When each equation is considered by itself, it represents the locus of the condition expressed in it, Arts. 14 and 20; and hence the operation of combining them to find the values of x and y is the algebraic method of finding the intersection of the loci of two conditions, which was done *graphically*, or by actually drawing the loci, in the example of Art. 15.

39. If, as in the above example, the equations are of the first degree, there is but one solution; the loci being straight lines which intersect in only one point, in this case the point (1, 2). But if one of the equations is of the second degree, and the other of the first, as $x^2 + y^2 = 25$, and $25 + x = 7y$, we shall have two solutions; for, substituting $x = 7y - 25$ in $x^2 + y^2 = 25$, we have $50y^2 - 350y + 625 = 25$, reducing to $y^2 - 7y + 12 = 0$, an equation for y of second degree, giving $y = 4$, or $y = 3$. The corresponding values of x, in $x = 7y - 25$ are $x = 3$ and $x = -4$: in fact, in this case, the loci are a circle and a straight line, which intersect in the two points, (3, 4) and (—4, 3). Both results may be verified by substitution in each of the equations.

Examples.—Find the intersection of $2y = 3x + 7$ and $y = 5 - \frac{1}{2}x$; of $x^2 + y^2 = 100$ and $x = 8$; of $x^2 + y^2 = 10$ and

$x = 3y$; of $x = y$ and $x = -y$; of $y - mx - b = 0$ and $my + x - a = 0$. Verify in each case.

40. When the equations are, in reality, *contradictory*, as $x + y = 1$, $x + y = 3$, *no* solution exists; the equations cannot be solved simultaneously, and the problem, supposed to give rise to them both, is *impossible*. The loci of such equations do *not* intersect at all; in the example, they are parallel straight lines. If we attempt to eliminate x between these equations, we have the impossible result, $0 = 2$. When one of the equations is of second degree, the impossibility of solution, or the fact that the loci do not intersect, is shown by the occurrence of imaginary quantities in the values of x and y. For instance, if the equations are $x^2 + y^2 = 25$ and $x = y - 8$, eliminating x we have $y^2 - 8y = -19\frac{1}{2}$, completing the square $(y - 4)^2 = -3\frac{1}{2}$, hence $y = 4 \pm \sqrt{-3\frac{1}{2}}$.

To assist himself in understanding this subject, the student may construct the loci, in the several examples given, both of intersection and non-intersection.

41. If there are points of intersection, and we *combine* the equations by addition or subtraction, we shall have an equation true of these points, whether we eliminate one of the unknown quantities or not. For the equations being both true of the common point or points, so also must be any equation derived from them. For instance, in our first illustration, if we subtract $x = 2y - 3$ from $x + y = 3$, member from member, we eliminate x; the result is $y = 6 - 2y$, giving at once the value of the ordinate, $y = 2$. But if we add them, we obtain $2x + y = 2y$ or $2x = y$, which is true of the point (1, 2), which we found to be the point of intersection. So also, if we multiply one of the equations through by any quantity, before combining; as $2x + 2y = 6$, to which if we add $x = 2y - 3$, the result reduces to $x = 1$, and from which if we subtract the same we have $x + 4y = 9$, which is still true of the point (1, 2). All of these equations, including $x = 1$ and $y = 2$ (which are the equations of parallels to the axes), are the equations of straight lines passing through the point of intersection of the original lines.

This may be generalized for equations of the first degree, thus: Let $Ax + By + C = 0$ and $A'x + B'y + C' = 0$ be the equations of any two lines, and k* any number positive or negative, then

$$Ax + By + C + k(A'x + B'y + C') = 0,$$

is the equation of a straight line passing through their intersection, whatever the value of k; for it is evidently satisfied by those values of x and y, which make both the expression, $Ax + By + C$ and $A'x + B'y + C'$ equal to zero, and it represents a *straight* line because it is of the first degree.

Examples.—Give the equations of a number of lines passing through the intersection of $2y = 3x - 7$ and $x + y = 9$.

Form the general equation of the straight line passing through this point, and give to k such values as to eliminate, successively, x and y.

42. General values of the co-ordinates of intersection may be found by giving k, in the above general equation, successively, such values as to eliminate y and x, thus making $k = -\frac{B}{B'}$ (or making the coefficients of y the same, and subtracting), to obtain the value of x; and making $k = -\frac{A}{A'}$, for y, we have

$$x = \frac{BC' - CB'}{AB' - BA'}, \quad \text{and} \quad y = \frac{A'C - C'A}{AB' - BA'},$$

for the general values of the co-ordinates of the point common to $Ax + By + C = 0$ and $A'x + B'y + C' = 0$, as may be verified by substituting these values for x and y in the two equations.

These values will both at the same time take the value infinity, when their common denominator is zero, that is, when $AB' = BA'$, or when $A : B :: A' : B'$. This indicates that the lines do not intersect, when the co-efficients of x and y in the two equations are proportional, as in the equations $2x + 3y + 5 = 0$ and $4x + 6y + 8 = 0$. When this is the case, the general equation above, which may be called the *equation of combination* of the two lines, represents a series of parallel lines instead of a series of lines passing

* The first letters of the alphabet a, b, etc., are used to denote lines or distances, as well as x, y, x', y', etc.; the letters k, l, m and n are used to denote abstract numbers or ratios. The capitals A, B, etc., are used to denote the coefficients in the general equations.

through a common point; as $2x + 3y + 5 + k(4x + 6y + 8) = 0$, for, being reduced to the form $y = mx + b$, its direction ratio is $-\frac{2+4k}{3+6k} = -\frac{2}{3}$, whatever the value of k.

Because in the case of parallel lines the general values of the co-ordinates of intersection become infinite, parallel lines are said to meet in a *point at infinity*. Every equation of the first degree may be considered as satisfied by infinite values of x and y, these infinite values having a certain *ratio to each other*, but a finite quantity having no ratio to one of them. For example, $y = 2x + 1$ is satisfied by infinite values of the variables, of which the value of y is double that of x; $y = 2x + 2$ is satisfied by the *same* infinite values, or passes through the *same* point at infinity.* The series of parallel lines represented by $2x + 3y + 5 + k(4x + 6y + 8) = 0$ may be considered as passing through a common point at infinity; and in general

$$Ax + By + C + k(A'x + B'y + C') = 0$$

represents a series of straight lines passing through a common point, even when the given lines are parallel.

43. If the lines whose equations are thus combined are parallel, a certain value of k will cause the terms containing x and y to disappear at once from the equation. In the numerical example above, this value is $k = -\frac{1}{2}$, which reduces the equation to $1 = 0$. For every value of k, except this, the equation is of the form $Ax + By + C = 0$, and therefore represents a straight line. But for this particular value of k, it takes the form $C = 0$, a constant equals zero, which is impossible. Now the various equations, produced by giving different values to k, represent lines passing through a certain "point at infinity;" hence the impossible equation of the form $C = 0$, which, strictly speaking, does *not* represent a line, because it can be satisfied by *no assignable* points, is called the equation of the *line at infinity*, that is, a line all of whose points are at infinity.

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* It must be remembered, however, that a "point at infinity" has no position, because no *assignable* co-ordinates. Its only definite property is the *ratio* conceived to exist between its co-ordinates. All expressions involving the idea of infinity are illogical, but convenient as adapting the language of the general case (intersection) to the special case (parallelism).

If in the equations $Ax + By + C = 0$, and $A'x + B'y + C' = 0$, $A : A' :: B : B' :: C : C'$; that is, if all the corresponding terms have the same ratio, the values of both co-ordinates of intersection take the form $\frac{0}{0}$, showing that the lines meet in every point, or coincide. For instance, the equations $2x + 3y + 5 = 0$ and $4x + 6y + 10 = 0$ represent the same line, since they are in fact the same equation, one being derived from the other by multiplying through by the same number.

Arbitrary Constants.

44. When the equation of a straight line is in the general form, $Ax + By + C = 0$, as $2x + 3y + 5 = 0$, it contains *three* constants; and the position of the line depends, not on their absolute values, but on their *ratios.*

By dividing through by one of the constants, the equation may be reduced to a form in which there are but *two* constants, whose *values* determine the position of the line. Two such forms have already been given, $y = mx + b$, and $x = ny + a$; the above equation reduced to the first of these forms, is $y = -\frac{2}{3}x - 1\frac{2}{3}$; reduced to the second, it is $x = -\frac{3}{2}y - 2\frac{1}{2}$.

45. The two constants, occurring in one of these algebraic forms are called *arbitrary constants*, because values may be given them at pleasure. The arbitrary constants in $y = mx + b$, are m and b, representing the direction ratio of the line, and its intercept on the axis of Y. If we wish to make the line fulfil certain conditions, we have to determine suitable values for the arbitrary constants. Thus, if we wish the equation of a line parallel to $2x + 3y + 5 = 0$, we must give to m, in $y = mx + b$, the same value that it has in the given line; namely, $-\frac{2}{3}$, because the line is to have the same direction. This gives $y = -\frac{2}{3}x + b$ for the equation; b is still arbitrary; for, whatever its value, the line fulfils the required condition of being parallel to $2x + 3y + 5 = 0$, or $y = -\frac{2}{3}x - 1\frac{2}{3}$. b may now be determined so as to make the line fulfil some other condition, for instance, that it be twice as far distant from the origin; in which case b must be twice as great as in the given line, giving $y = -\frac{2}{3}x - 3\frac{1}{3}$ or $2x + 3y + 10 = 0$. This last result might have been obtained by simply doubling the absolute term in

the given equation, which it is easy to see has the effect of doubling the intercepts.

Examples.—Give the equations of lines parallel to $x + 2y = 5$. 1st, making the intercept 1 on the axis of Y; 2d, twice as far from the origin; 3d, passing through the origin; 4th, making the intercept 3 on the axis of X.

For the last case assume the form $x = ny + a$, and determine n to be the same as in the given line, and $a = 3$.

46. When the axes are *rectangular*, the direction ratio, m, is a simple trigonometric function of the inclination of the line to the axis of X; for PRO, fig. Art. 30, being a right angle, $m = \frac{PR}{OR} =$ tan POR, the *tangent* of the line's inclination. In the figure of Art. 32, $m = \frac{P'R}{OR} =$ tan P′OR, which is the same as the inclination of PB to the axis of X. n, which is the reciprocal of m, is the *cotangent* of the inclination to the same axis. Hence, when the axes are rectangular, we can find the inclination of a given line to the axis of X by means of the trigonometric tables. Thus $y = 2x + 3$ makes an angle with the axis of X, whose tangent is 2, because $m = 2$. By the use of 5-place tables, the logarithm of 2 is 0.30103, and the angle corresponding to this as a logarithmic tangent is $63° \ 26' \ 6''$, which is therefore the inclination of this line to the axis of X. The inclination of $y = x + 2$ in which $m = 1$ is 45°, because the tangent of 45° is unity. If m is negative, the angle is obtuse, or in the second quadrant, as in the line $2x + 3y + 5 = 0$, where $m = -\frac{2}{3} = -0.66667$, and the angle will be found to be $146° \ 18' \ 36''$.

Examples.—The axes being rectangular, what is the inclination of $y = 3x - 2$ to the axis of X? of $x - 2y + 3 = 0$? of $x + y = 0$? the mutual inclination or difference of inclinations of $x - 3y + 5 = 0$ and $y = 3x + 2$?

47. If the equation of a line making a given angle with the axis of X be required, we may assume $y = mx + b$ and compute the value of m from the given angle. If it is to make a given angle with a given line, for instance, 30° with the line $y = 2x + 3$, we ascertain the inclination of the given line, as above, to be $63° \ 26' \ 6''$, and add or subtract 30°, according as we require the inclina-

tion of the new line to exceed or fall short of that of the given one. Suppose it required to exceed it by 30°; then the inclination is 93° 26′ 6″, whose natural tangent is — 16.66. Hence the equation is approximately $y = -16.66\,x + b$, in which b is still arbitary, as explained in Art. 45.*

48. If the line is to be perpendicular to the given line, it is not necessary to compute its inclination; for suppose a to represent the inclination of the given line, then $\tan a$ is known, and the value of m required is $\tan(90° + a)$. Now, $\tan(90° + a) = -\tan(90° - a) = -\cot a = -\dfrac{1}{\tan a}$, because $90° - a$ is the supplement of $90° + a$ and the complement of a. Hence for a perpendicular line, take for m the *negative of the reciprocal* of m in the given line; thus given $y = 3x - 2$, in which $m = 3$, for a perpendicular line we must have $m = -\frac{1}{3}$, giving the equation $y = -\frac{1}{3}x + b$. The condition that two given lines should be perpendicular is, that the values of m should be of opposite signs and reciprocals, thus $y = x + 3$ and $y = 3 - x$ are perpendicular.

EXAMPLES.—Give the equations of a number of lines perpendicular to $3x + y = 5$; of a line passing through the origin and perpendicular to $7y - 6x + 3 = 0$.

Of the lines $3x + 2y + 1 = 0$, $3x - 2y + 1 = 0$, $2x + 3y + 1 = 0$, $2x - 3y + 1 = 0$, which are perpendicular?

It must be remembered that the results of the last three Articles do not apply to oblique co-ordinates, in which case m is not a simple function of the inclination of the line to either axis, but depends upon its inclination to both axes, and hence upon the angle between them.†

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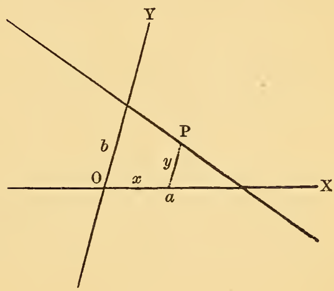
* When the angle between two given lines is required, we may with advantage use the formula for the tangent of the difference of two angles, $\tan(\theta - \theta') = \dfrac{\tan\theta - \tan\theta'}{1 + \tan\theta\tan\theta'}$; thus, given $y = 2x - 3$ and $y = x + 2$ $\tan\theta = 2$, $\tan\theta' = 1$, hence $\tan(\theta - \theta) = \frac{1}{3}$. When a line making given angle with given line is required the method in the text is the simplest in computation.

† In general, let a = POR, figure of Art. 30, or the line's inclination to the axis of X, β = POY = OPR, and ω = YOX, then by trigonometry $m = \dfrac{\mathrm{PR}}{\mathrm{OR}} = \dfrac{\sin \mathrm{POR}}{\sin \mathrm{OPR}} = \dfrac{\sin a}{\sin\beta}$, and $a + \beta = \omega$. Hence $m = \dfrac{\sin a}{\sin(\omega - a)}$,

Forms of the Equation of the Straight Line.

49. In finding the algebraic equation of the straight line, we may employ any constants whose values determine the position of the line. In order that the equation should be capable of representing *any* line, there must be *at least two* constants; an equation containing but one can only represent a particular class of lines; as $y = mx$, lines passing through the origin; $x = a$, lines parallel to the axis of Y. In $y = mx + b$, the two constants are the abstract number m, or direction ratio and the length, or number of linear units b, which is the intercept on the axis of Y. In $x = ny + a$, they are the reciprocal of the direction ratio and the intercept on the axis of X.



50. Let us now find the equation of the straight line in terms of its two intercepts. In conformity with the notation of the previous equations let a represent the intercept on the axis of X, and b on that of Y; then drawing the ordinate of any point, P, we have, by similar triangles, $b : y :: a : a - x$, or $\frac{y}{b} = \frac{a - x}{a} = 1 - \frac{x}{a}$.

Hence $$\frac{x}{a} + \frac{y}{b} = 1$$

is the relation existing between the constants a and b and the co-ordinates of any point of the line. This is the equation of a line in terms of its intercepts; for let $y = 0$, and we have $x_0 = a$; let $x = 0$, and we have $y_0 = b$. If x be taken negative, in the figure, where a and b are positive, y is greater than b and $\frac{y}{b} > 1$, but $\frac{x}{a}$ is then negative; so that the algebraic sum of the quotients $\frac{x}{a}$ and $\frac{y}{b}$ is still unity. If a is negative, as it would be for the line

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which renders the process of finding α from m much more difficult. When $\omega = 90°$ this value of m reduces to tan α.

BP, figure Art. 32, $\frac{x}{a}$ is negative and $y > b$, for positive values of x. By means of this formula we can give at once the equation of a line making given intercepts; thus, required that $x_0 = 7$ and $y_0 = 5$, the equation of the line is $\frac{x}{7} + \frac{y}{5} = 1$, or $5x + 7y = 35$; required $x_0 = -2$, $y_0 = 2$ the equation is $-\frac{x}{2} + \frac{y}{2} = 1$, or $y - x = 2$. If the line is parallel to one axis, for instance the axis of X, the intercept on that axis is infinite, and the fraction $\frac{x}{a} = 0$, the equation therefore reduces to $\frac{y}{b} = 1$ or $y = b$, as before found. If the line passes through the origin, the intercepts are both zero, and the fractions become infinite, so that the equation of this class of lines cannot be put in the form $\frac{x}{a} + \frac{y}{b} = 1$

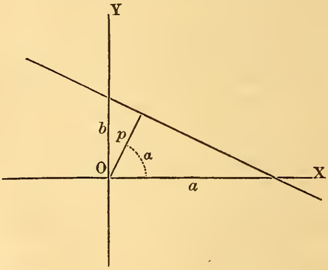
EXAMPLES.—Give the equations of lines making the intercepts $x_0 = 1$, $y_0 = -2$; $x_0 = -5$, $y_0 = 10$; $x_0 = -1$, $y_0 = \infty$, etc., etc.

In each case reduce the line to the form $y = mx + b$, and give its direction ratio.

Find a general expression for the direction ratio of $\frac{x}{a} + \frac{y}{b} = 1$.

Ans. $m = -\frac{b}{a}$.

51. When the axes are rectangular, an especially convenient form of the equation of the straight line is that in which the constants are the length of the perpendicular from the origin on the line, and the inclination of this perpendicular to the axis of X. In the figure, p represents the perpendicular and α its inclination and a and b the intercepts. If now we find values for a and b, in terms of p and α, and substitute them in $\frac{x}{a} + \frac{y}{b} = 1$, we shall have the required equation.



By the definitions of the trigonometric functions $\frac{p}{a} = \cos\alpha$ and $\frac{p}{b} = \sin\alpha$. Hence the equation,

$$x\cos\alpha + y\sin\alpha = p.$$

In this form the arbitrary constants are a distance p, and angle α, whose functions $\cos\alpha$ and $\sin\alpha$ are ratios or abstract numbers. In the figure, α is in the first quadrant or $< 90°$. If α were in the second quadrant, it is evident the line would make a positive intercept on the axis of Y, and a negative one on the axis of X, in which case, $\cos\alpha$ would be negative and $\sin\alpha$ positive. If α were in the third quadrant, both $\cos\alpha$ and $\sin\alpha$, and both intercepts, would be negative; if in the fourth quadrant, $\cos\alpha$ is positive and $\sin\alpha$ negative, hence x_0 is positive and y_0 negative. Hence we see that with a proper value of α, admitting angles in all the four quadrants, the equation of a line in any position may be expressed in this form, the value of p being considered always positive. Thus, if we require the equation of a line seven units distant from the origin, we have $x\cos\alpha + y\sin\alpha = 7$, which by giving different values to α, may be made the equation of any straight line at that distance from the origin. For instance, $\alpha = 0°$ gives the equation $x = 7$, $\alpha = 90°$ gives $y = 7$, $\alpha = 180°$ gives $-x = 7$ or $x = -7$, $\alpha = 270°$ gives $-y = 7$ or $y = -7$, $\alpha = 45°$ gives $\sqrt{\frac{1}{2}}\,x + \sqrt{\frac{1}{2}}\,y = 7$, or $x + y = 7\sqrt{2}$, $\alpha = 135°$ gives $-\sqrt{\frac{1}{2}}\,x + \sqrt{\frac{1}{2}}\,y = 7$, or $y - x = 7\sqrt{2}$.

52. The equation of any straight line can be written in this form, in which the absolute term is the perpendicular distance of the line from the origin. To put an equation in this form, it is not necessary to find the value of the angle α, but only to make the coefficients of x and y in the equation the sine and cosine of *some* angle. Now the sum of the squares of the sine and cosine of any angle is unity. If then the coefficients have this property, the equation is in the form $x\cos\alpha + y\sin\alpha = p$, though the value of α is not stated; thus $\frac{3}{5}x - \frac{4}{5}y = 3$ is in the above form. If the sum of the squares of the coefficients is not unity, we must divide the equation through by the square root of this sum: thus, given $12y - 5x + 26 = 0$, we find the sum of the squares to be 169; we therefore divide the equation through by $\sqrt{169}$ or 13, which gives

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$\frac{12}{13}y - \frac{5}{13}x + 2 = 0$, or $\frac{5}{13}x - \frac{12}{13}y = 2$, which is in the required form, and shows that the perpendicular from the origin on the line is two units in length. The inclination of this perpendicular is the angle whose cosine is $\frac{5}{13}$, and whose sine is $-\frac{12}{13}$, an angle evidently in the fourth quadrant, and whose value can be found from the trigonometric tables. Usually the divisor will be a surd; as for $x + y = 2$, which reduced is $\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \sqrt{2}$, the perpendicular being $\sqrt{2}$.

Examples.—The axes being rectangular, what is the distance of the line $3x - y = 4$ from the origin? of the line $4x - 3y = 10$? of the line $7x = 21$? etc., etc.

53. A general expression for the distance of a line from the origin may be found by reducing the general equation $Ax + By + C = 0$ to this form, which gives

$$\frac{A}{\sqrt{A^2 + B^2}}x + \frac{B}{\sqrt{A^2 + B^2}}y = \frac{-C}{\sqrt{A^2 + B^2}},$$

the distance from the origin is therefore numerically $\frac{C}{\sqrt{A^2 + B^2}}$, or the absolute term divided by the square root of the sum of the squares of the coefficients. We cannot tell whether we ought to change the signs of the above equation throughout, unless we know the sign of C.

The equations $x = a$ and $y = b$ are already in this form; accordingly a and b are perpendiculars from the origin upon these lines, when the axes are rectangular.

54. If in the equation $x \cos \alpha + y \sin \alpha = p$, we make $p = 0$, we have a parallel line passing through the origin; if we make p negative, the effect is the same as if we had changed the sign of both $\cos \alpha$ and $\sin \alpha$, in the first member, which is equivalent to replacing α by $\alpha + 180°$, or measuring the perpendicular in a directly opposite direction. All lines having the same value of α, or values differing by 180°, are parallel, but to ascertain in which direction the perpendicular is measured, we must make the second member of the equation positive, and find the corresponding value of α. Thus $\frac{12}{13}x - \frac{5}{13}y = 1$, $\frac{12}{13}x - \frac{5}{13}y = 0$, and $\frac{12}{13}x - \frac{5}{13}y = -1$,

are parallel; but in the first case $a = 337° \; 22' \; 48''$, in the fourth quadrant, because cos a is positive, and sin a is negative, and in the third case $a = 157° \; 22' \; 48''$, in the second quadrant; for changing signs throughout to make the second member positive, we see that cos a is negative, and sin a positive. In the second equation it is immaterial which value we give to a.

EXAMPLES.—Give the value of a, in $3x - y = 4$, $3x = y$, $3x - y = -4$; in $x + y = 1$, $x + y = -1$; in $7x = 21$, $7x = -21$, etc., etc.

EQUATIONS OF CONDITION.

55. In preceding articles we have found several algebraic forms of the equation of the straight line, in which, by directly giving proper values to the constants, we produce the equations of lines fulfilling certain conditions: we shall now give a general method, by which the constants in an equation may be determined so as to make the line pass through given points. Suppose, for instance, it is required to find the equation of a straight line passing through the point (7, 5). Assume the equation $y = mx + b$, which we know to be the equation of a straight line, whatever the values of m and b, the proper values of these constants being, however, as yet unknown. Now, by the condition, the point (7, 5) must satisfy the equation; hence $5 = 7m + b$. Whenever m and b are so related that this equation is true, the given point satisfies the equation; or the equation represents a line passing through the point. It therefore imposes a *condition* upon m and b, equivalent to the condition proposed in the problem. As there are two unknown quantities, m and b, to be determined, we can satisfy two such *equations of condition.* Thus, let the line be required also to pass through the point (2, — 7); the equation expressing this condition is $-7 = 2m + b$. Solving these two equations of condition, we have $m = 2\frac{2}{5}$, $b = -11\frac{4}{5}$, which are the only values of m and b which will satisfy them both. These are therefore the proper values of m and b in the assumed equation $y = mx + b$; substituting them, we have $y = 2\frac{2}{5}x - 11\frac{4}{5}$, or $5y = 12x - 59$. The result may now be verified by substituting the co-ordinates of the given points for x and y, and showing that each point satisfies the equation.

EXAMPLES.—Find the equation of the straight line passing through $(-1, -2)$ and $(0, -3)$; through the origin and $(1, -1,)$; etc., etc.

Find the equation of the line making the intercepts $x_0 = 6$, $y_0 = 8$, as the line passing through $(6, 0)$ and $(0, 8)$.

56. We can thus find the equation of a straight line fulfilling two conditions, or passing through two given points, because there are two arbitrary constants to be determined, which allows us to satisfy two equations of condition and no more. If we should assume the form $Ax + By + C = 0$, and attempt to satisfy three equations of condition, we should find $A = 0$, $B = 0$ and $C = 0$. In general we can satisfy as many equations of condition, and hence make a line pass through as many points, as there are constants in the assumed equation, but we must assume the equation in such a form that one of the terms contains no constant, otherwise every equation of condition will be satisfied by making all the constants zero. Thus, $Ax^2 + Dx + Ey + F = 0$ is the general equation of a certain class of curves. If, however, we wish to find that curve of the class which passes through certain given points, we must assume the form $x^2 + Dx + Ey + F = 0$, and then it is evident, that three equations of condition may be satisfied, or the curve may be made to pass through three given points.

Formulæ for Straight Lines.

57. The two constants in either of the algebraic forms, $y = mx + b$, $x = ny + a$, $\frac{x}{a} + \frac{y}{b} = 1$, and $x \cos a + y \sin a = p$, are quantities having particular values for a given straight line, which values may be ascertained by reducing the equation to the proper form: thus we reduce the equation to the form $y = mx + b$ to ascertain the line's direction ratio; to the form $x \cos a + y \sin a = p$ to find its distance from the origin.

These equations are also formulæ, by which the equation of a line is expressed in terms of certain constants whose values may be given, as in the examples of Art. 50, or found by an algebraic process as in Art. 55.

58. The equation of the straight line may also be expressed in terms of the co-ordinates of known points on the line, and then we shall have formulæ for lines passing through given points. This

may be done by means of the method of equations of condition, explained in Art. 55.

Suppose, in the first place, the co-ordinates of P′, a point of the line, to be known, or the line required to pass through a known point (x', y'). The equation $y = mx + b$ being assumed, we have, because P′ is a point of the line, the equation of condition,

$$y' = mx' + b.$$

By means of this we can eliminate from the equation of the line one of the arbitrary constants, introducing in its stead the known quantities x' and y'. Thus, substituting for b in $y = mx + b$ its value, $b = y' - mx'$, from the equation of condition, we have $y = mx + y' - mx'$, or

$$y - y' = m(x - x'),$$

in which the arbitrary constant m is retained.

This is a formula for a straight line passing through a given point. It still contains an arbitrary constant, because there is a variety of lines passing through a given point. In fact, whatever the value of m, the line evidently passes through P′, because substituting x' and y' for x and y both members reduce to zero, independently of the value of m. By means of a proper determination of m, the line may be made parallel to a given line, or, the axes being rectangular, perpendicular to a given line. Thus, for the line passing through (7, 5), substituting $y' = 5$ and $x' = 7$, we have $y - 5 = m(x - 7)$. If the line is also to be parallel to $2y - x + 3 = 0$ or $y = \frac{1}{2}x - \frac{3}{2}$, we make $m = \frac{1}{2}$, giving $y - 5 = \frac{1}{2}(x - 7)$, or $2y = x + 3$. If it is to be perpendicular to this line, we make $m = -2$, giving $y - 5 = -2(x - 7)$, or $y = 19 - 2x$. The results are verified by showing that they are satisfied by the values $x = 7$, $y = 5$.

Examples. — Find the equations of lines passing through $(-1, 2)$: 1st, parallel; 2d, perpendicular to $3y + 2x = 5$: through (0, 6) parallel to $y + x = 0$, etc., etc.

What is the general formula for a line passing through P′ and perpendicular to $y = mx + b$?

Ans. $y - y' = -\dfrac{1}{m}(x - x')$.

59. The formula $y - y' = m(x - x')$ is a more general one,

including the previous ones in which m occurs; for, make $x' = 0$ and $y' = 0$, and it reduces to $y = mx$, the equation of a line passing through the origin; make $x' = 0$ and $y' = b$, the co-ordinates of B in the figure of Art. 32, and it reduces to $y - b = mx$, or $y = mx + b$, the equation of a line making the intercept b on that axis. On the other hand, it is a special case of the still more general formula of Art. 41, formed by combining the equations of two lines in their general form,

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C} + k(\mathrm{A}'x + \mathrm{B}'y + \mathrm{C}') = 0.$$

For $x = x'$ and $y = y'$ are the equations to two straight lines passing through P′; namely, the lines parallel to the axes. Putting them in the forms $x - x' = 0$, $y - y' = 0$, and combining, we have $y - y' = -k(x - x')$, or, writing the letter m instead of $-k$ for the arbitrary constant, $y - y' = m(x - x')$. So $y = mx$ may be derived by combining the equations of the axes, $y = 0$ and $x = 0$, and is therefore the general equation of a line passing through their intersection, the origin.

60. The general equation of combination above, as well as the present formula, which is a special case of it, contains one arbitrary constant, because it represents a line fulfilling one condition—namely, that of passing through a particular point; but not *determined* by that condition, inasmuch as a straight line can be made to fulfil *two* conditions. That constant can now be determined so as to make the line fulfil any other condition required. For example, required the equation of a line passing through the intersection of $2y = 3x - 7$ and $x + y = 9$, and parallel to $y = 2x + 3$. We may use the method of combining the equations to avoid the necessity of finding the intersection. Thus, multiplying the members of $x + y = 9$ by k, and adding to those of $2y = 3x - 7$ (it is not essential to transpose all the terms to one member), we have $2y + kx + ky = 3x - 7 + 9k$; reducing this to the form $y = mx + b$, it becomes $y = \dfrac{3-k}{2+k}x + \dfrac{9k-7}{2+k}$. Now, to be parallel to $y = 2x + 3$ we must have the direction ratio $\dfrac{3-k}{2+k} = 2$. This determines the value of k; for clearing of fractions, $3 - k = 4 + 2k$, or $k = -\frac{1}{3}$. This value we may now substitute in the

original equation, or in the value of b in the reduced equation, which gives $b = \frac{9k - 7}{2 + k} = -6$, hence the equation required is $y = 2x - 6$. k may also readily be determined, so as to make the line pass through a given point. Suppose the above line $2y + kx + ky = 3x - 7 + 9k$ required to pass through (1, 2). The equation of condition is $4 + k + 2k = 3 - 7 + 9k$, hence $k = 1\frac{1}{3}$, substituting which, we have for the equation of the line $2y + 1\frac{1}{3}x + 1\frac{1}{3}y = 3x - 7 + 12$, or $\frac{10}{3}y = \frac{5}{3}x + 5$, or $2y = x + 3$. This is verified for this last condition by substituting 1 and 2 for x and y; and the first condition—that it pass through the intersection of $2y = 3x - 7$ and $y + y = 9$—may be verified by finding that intersection, which is (5, 4), and substituting in $2y = x + 3$. The equation $y + 2x - 6$ is also satisfied by (5, 4), hence both the lines found pass through the intersection of the given lines.

Examples.—Give the equation of the lines passing through the intersection of $2y = 3 - 4x$ and $4 - y = 5x$: 1st, parallel; 2d, perpendicular to $3y = 2x + 6$; through the intersection of $x = 5$ and $x - y = 9$: 1st, parallel to $x + y = 0$; 2d, passing through (2, 1); of $2y + x = 5$, and $3 - y = x$: 1st, parallel to $x - y = 10$; 2d, passing through $(-3, 5)$, and verify.

61. Every equation of first degree containing only one arbitrary constant may be regarded as a case of $Ax + By + C + k(A'x + B'y + C') = 0$, and therefore represents a series of lines passing through a single point. This point is the intersection of two straight lines whose equations consist, one of those terms which do not contain the arbitrary constant, and the other, of those which do. Thus $x + 2y_1x + y_1y = 25$, where y_1 is an arbitrary constant, is the equation of a line passing through the intersection of $x = 25$ and $2x + y = 0$, that is, through the point $(25, -50)$; for, in $x - 25 + k(2x + y) = 0$, we have only to put y_1 in place of k to produce the given equation. Therefore whatever the value of y_1 the line passes through this point. This is verified by showing that $x = 25$, $y = -50$ satisfies the equation independently of y_1.

Examples.—Through what point does the line $l(x - y) + 3x + 2y = 2 + l$, necessarily pass?

Ans. The intersection of $3x + 2y = 2$ with $x - y = 1$, which is the point $(\frac{4}{5}, -\frac{1}{5})$.

Verify this result for a number of assumed values of l, as when $l = 1$, when $l = 2$, etc.

Find the point common to the lines represented by $2cx - (y - c) = 3c\,(y + x) + 4\,(x - 1)$.

62. The most ready way of finding the co-ordinates of the point sought in the above examples, is to give the arbitrary constants such values as to eliminate successively x and y from the equation; thus finding the equations of two lines of the series—namely, those parallel to the axes, or of the forms $x = a$ and $y = b$.

For instance, in $l\,(x - y) + 3x + 2y = 2 + l$, we eliminate y by making $l = 2$, which gives $5x = 4$ or $x = \frac{4}{5}$, the equation of a line parallel to the axis of Y passing through the required point, hence $\frac{4}{5}$ is the abscissa of that point. Similarly, $l = -3$ gives an equation of the form $y = b$, $5y = -1$ or $y = -\frac{1}{5}$. This is in fact the same process as that used in Art. 42 to find general values of the co-ordinates of intersection. By giving l in the example the value -2, we shall have an equation of the form $y = ax$, because $l = -2$ makes the absolute term disappear. The result $x + 4y = 0$ is therefore the equation of the line of the series passing through the origin. In general, giving k the value $-\frac{C}{C'}$, we have for the equation of the line passing through the intersection of $Ax + By + C = 0$ with $A'x + B'y + C' = 0$, and the origin,

$$C'\,(Ax + By + C) - C\,(A'x + B'y + C') = 0.$$

The value $k = -\frac{C}{C'}$ might have been found by the condition that the origin should satisfy the equation.

63. If the two equations into which we decompose an equation with one arbitrary constant, are the equations of parallel lines, as in the example, $l\,(x - y) + 3x - 3y = 10$, the point common to the system of lines is at infinity; that is, the given equation represents a series of parallel lines; in the example, all parallel to $x - y = 0$. So too, if one of the equations is of the form $C = 0$ (a constant equal zero), the impossible equation, which in Art. 43 was interpreted as the equation of the line at infinity; as $l\,(x - y) = 1$, which represents the same series of parallel lines, all parallel lines meeting the line at infinity in the same point. The equation

$y = mx + b$, when b alone is considered as an *arbitrary* constant, represents a series of lines passing through a common point at infinity, the point in which $y = mx$ meets the line at infinity.

EXAMPLES.—What series of lines is represented by $l(x-2) + l(y-x) = (l+2)y + 3x - 1$?

What, by the general equation $Ax + By + C = 0$ when B and C are fixed, and A is an arbitrary constant.

Ans. A series of straight lines passing through the intersection of $x = 0$ and $By + C = 0$, or the point $\left(0, -\frac{B}{C}\right)$.

What, when B alone is arbitrary? and what when C alone is arbitrary?

64. The equation of a line passing through a given point, $y - y' = m(x - x')$, contains one arbitrary constant; by means of an equation of condition this constant may be determined so as to make the line pass through another given point. Let P'' be this second point, then

$$y'' - y' = m(x'' - x'), \quad \text{or} \quad m = \frac{y'' - y'}{x'' - x'}$$

is the condition that P'' should be a point of the line; substituting this value of m in $y - y' = m(x - x')$, we have

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'),$$

which is the formula for the straight line passing through P' and P''. In making use of this formula, the co-ordinates of one of the given points are substituted for x' and y', and those of the other for x'' and y'': thus required the line passing through $(-1, 2)$ and $(3, -1)$; calling the first point P', we have $y - 2 = \frac{-3}{4}(x + 1)$, which reduces to $4y + 3x = 5$. The result is verified by showing that each given point satisfies the equation. If the second point had been taken as P' and $(-1, 2)$ as P'', we should have $y + 1 = \frac{3}{-4}(x - 3)$, reducing also to $4y + 3x = 5$.

In applying this formula it is best first to find the value of the direction ratio $\frac{y'' - y'}{x'' - x'}$. Two special cases may occur: when the ordinates of the two points are the same, the direction ratio becomes zero and the equation reduces to $y - y' = 0$ or $y = y'$; and when the abscissas are equal, the direction ratio is infinite and the equation of the line is $x = x'$. Thus, the line passing through (1, 2) and (— 3, 2) is $y = 2$; that passing through (1, 2) and (1, — 1) is $x = 1$. In the latter case the formula $y - y' = \frac{y'' - y'}{x'' - x'}(x - x')$ may be regarded as expressing that y is indeterminate, for $x - x'$ always equals zero, when $\frac{y'' - y'}{x'' - x'}$ is infinite, and $0 \times \infty$, like $\frac{0}{0}$, is indeterminate in value.

EXAMPLES.—Give the equation of the line joining (— 2, 0) and (1, — 1): passing through (6, 3) and (2, — 1), etc., etc.

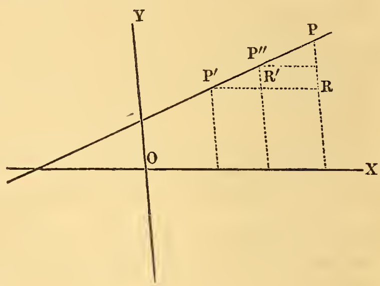
The equations of the sides of the triangle whose vertices are (1, 2) (2, 3) and (3, 1).

The sides of the triangle whose vertices are (6, 2), (—1 —8) and (— 3, 0).

Find the equation of the line joining $(a, 0)$ and $(0, b)$.

Show that the point found in Art. 11 satisfies the equation of P′ P″.

65. As the formulæ for lines passing through one or two given points are very important, we give also, the method of deriving them geometrically, that is, directly from a figure. Let P′ and P″ be the given points, and P any point of the line. Draw the ordinates of these three points, and parallels to the axis of X through P′ and P″: then $PR = y - y'$ and $P'R = x - x'$. It is evident that wherever on the line P be taken, PR and P′R will have the same ratio, which is the direction ratio of the line, and may be denoted by m, hence $\frac{y - y'}{x - x'} = m$, or



$$y - y' = m(x - x').$$

But $$\frac{P''R'}{P'R'} = \frac{y'' - y'}{x'' - x'} = m, \text{ hence}$$

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x').$$

This equation may also be derived from the similar triangles PRP′, P″R′P′, which give the proportion $y - y' : x - x' :: y'' - y' : x'' - x'$.

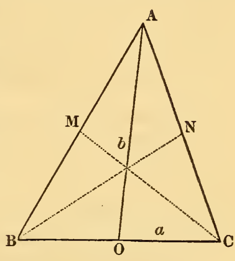
For a line passing through a given point and the origin, make $x' = 0, y' = 0$, and we have $y = \frac{y''}{x''}x$, or writing x', y' in place of x'', y'', since there is but one given point,

$$y = \frac{y'}{x'}x, \text{ or } x'y = y'x.$$

Thus, the line joining (7, 5) to the origin is $7y = 5x$.

Demonstration of Geometrical Theorems.

66. The analytic method is well adapted to the demonstration of geometrical theorems in which it is proved that certain lines pass through a common point. For instance, to prove that the three lines joining the vertices of a triangle with the middle points of the opposite sides pass through a common point. Let ABC be any triangle, and MNO the middle points of its sides. We simplify the demonstration by taking as axes the lines AO and BC; then we have to form the equations of MC and BN, and prove that they cut the axis of Y —the line AO—in the same point. Denote the distance AO by b, and OC by a.



Then the co-ordinates of the middle point N, between $A(0, b)$ and $C(a, 0)$, are $\frac{1}{2}a$ and $\frac{1}{2}b$; those of M are $-\frac{1}{2}a$ and $\frac{1}{2}b$. Using the formula for a line passing through two points, the equation of BN passing through B $(x' = -a, y' = 0)$, and N $(x'' = \frac{1}{2}a, y'' = \frac{1}{2}b)$ is $y = \frac{-\frac{1}{2}b}{-\frac{3}{2}a}(x + a)$. To find where this line cuts AO, the axis of Y, make $x = 0$; the result is $y_0 = \frac{1}{3}b$. Similarly, the equation of CM is $y = \frac{-\frac{1}{2}b}{\frac{3}{2}a}(x - a)$, in which also $y_0 = \frac{1}{3}b$; hence BN and

CM cut AO in the same point, each cutting off one-third of its length. Thus we have not only demonstrated the existence of a common point, but found its position on the line AO.

67. The three lines here proved to meet in a point are called the *bisectors* of the sides of the triangle. By means of the formulæ of Art. 11 for the middle point of any line, and the formula for a line passing through two points, we may form the equations of the bisectors for any given triangle, and then find the point where two of them intersect, and show that it satisfies the equation of the third. Thus, in the triangle whose vertices are $(1, 6)$ $(-3, 2)$ and $(5, -1)$, the middle point of the first side is $(-1, 4)$, the line joining this with $(5, -1)$ is found to be $6y + 5x = 19$. Similarly the equations of the other two are $12y = x + 27$, and $x = 1$: these meet in the point $(1, 2\frac{1}{3})$ and this point satisfies $6y + 5x = 19$. If we let P_1, P_2, P_3 represent three given points, we might by this method find formulæ for the bisectors and the co-ordinates of the point where they meet, in terms of the vertices of the triangle, and give a general proof that the bisectors of any triangle meet in a point.* As a method of proof, it would be much less simple than the one we have used, which is perfectly *general* as a demonstration, because ABC was *any* triangle, although for the sake of simplicity the axes were chosen in a particular way.

EXAMPLES.—Find the bisectors of the triangle $(1, 2)$ $(2, 3)$ and $(-3, 1)$, and show that they meet in a point.

Find the bisectors of the triangle $(6, 2)$ $(-1, -8)$ and $(3, 0)$, and the point in which they meet.

Prove the theorem, taking two sides of the triangle as axes.

Find the equations of perpendiculars to the sides of the above triangles at their middle points, and show that these lines also meet in a point, the axes being supposed rectangular.

Prove that these perpendiculars in any triangle meet in a point, taking as axes one of the sides and its perpendicular.

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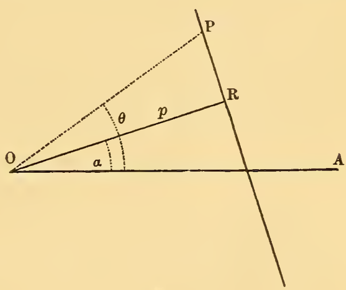
* The equation of the bisector of the side $P_2 P_3$ is $(y - y_1)(x_2 + x_3 - 2x_1) = (y_2 + y_3 - 2y_1)(x - x_1)$, from which, by interchanging the subscript numbers, may be formed the other equations. The point $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$ is the common point, for it satisfies the above equation and by symmetry must satisfy the others.

Find the equations of perpendiculars drawn from the vertices of the above triangles, to their opposite sides, and show that they also meet in a point.

Prove that in any triangle these perpendiculars meet in a point, taking one of the sides and its perpendicular as axes.

Polar Equation of Straight Line.

68. The polar equation of a straight line is the relation which exists between the polar co-ordinates of each of its points. If, however, the line passes through the pole, as in the figure of Art. 28, θ is constant for all points of the line, at least, when negative values of r are admitted, thus $\theta = 100°$, $\theta = 40°$, and in general $\theta = a$ where a stands for the line's inclination, represents a line passing through the pole. In all other cases θ is variable, and we must find a relation between the co-ordinates of a point of the line, and certain constants. Let O be the pole, OA the initial line, and PR any straight line: let us use the same constants as in the rectangular equation of Art. 51—namely, p, the length of the perpendicular, OR, from the pole, and a its inclination to the initial line. Draw OP the radius vector of any point of the line; then ORP is a right-angled triangle, of which the angle POR is represented by $\theta - a$, r is the hypothenuse, and OR the side adjacent to this angle; hence by the definitions of the trigonometrical functions



$\cos(\theta - a) = \frac{p}{r}$, or

$$r \cos(\theta - a) = p.$$

69. This is the relation between r, θ, p and a, and hence is the equation of the line PR. The constants, p and a of this equation, are the polar co-ordinates of R, the point of the line nearest the pole; accordingly if we let $\theta = a$, ($\theta - a$ being zero, whose cosine is unity), we have $r = p$. If we take P, in the figure, below the point R, $\theta - a$ is negative, and $< 90°$, or in the fourth quadrant, but

its cosine is still positive, and r is positive. Since $r = \dfrac{p}{\cos(\theta - a)}$, the smallest value of r corresponds to the greatest value of the cosine, which is given by $\theta = a$; as θ increases the cosine decreases, and r increases, and when $\theta = 90° + a$, $\cos(\theta - a) = 0$, and r is infinite, that is, there is no point of the line in a direction from the pole 90° in advance of that of OR. When θ exceeds this value, r is negative, in which case the line, drawn in the direction θ, *diverges* from the line PR, but being produced backward through the pole would intersect it. When $\theta = 270° + a$, the value of r is again infinite; and when it exceeds this value, r is positive. Thus during a complete revolution of the radius vector, its extremity P describes the line *twice;* since values of θ, differing by 180°, evidently give the same values of r with opposite signs, and therefore lead to the same point. During succeeding revolutions the line is described again, as values of θ differing by 360° give the same values of r.

70. The constants, p and a, which determine the position of the line, being also the polar co-ordinates which determine the position of R, $r \cos(\theta - a) = p$ is a formula for the equation of a line whose nearest point is given. Thus, the co-ordinates of the nearest point being (7, 100°), the equation is $r \cos(\theta - 100°) = 7$. If we use the co-ordinates (—7, 280°), which denote the same point, as constants, we shall have $r \cos(\theta - 280°) = -7$, which is, in fact, the same equation, the signs of both members having been changed at once.

Thus we interpret equations in which p is negative; when such is the case, we can make p positive by adding 180° to, or subtracting it from, the value of a. If $p = 0$, the equation reduces to $r \cos(\theta - a) = 0$. This is satisfied by $r = 0$, and any value of θ, that is, by the pole, but if r has any other value than zero $\cos(\theta - a) = 0$, hence $\theta - a = 90°$ or $270°$, that is, $\theta = a + 90°$ or $\theta = a + 270°$. Now $\theta = a$ is the equation of the straight line OR passing through the pole and making the inclination a, and either of the above is the equation of a line passing through the pole and perpendicular to OR; that is, parallel to PR. Hence all lines having the same value of a, or values differing by 180°, their equations being in the form $r \cos(\theta - a) = p$, are parallel to one another and perpendicular to the line $\theta = a$.

71. By giving the proper value to a, the line may be made to have any given direction. If $a = 0$, it is perpendicular to the initial line, and the equation reduces to

$$r \cos \theta = p,$$

in which, if p is positive, the line cuts the initial line on the right of the pole, but if p is negative, it cuts the initial line produced to the left of the pole. In the latter case we might make p positive and $a = 180°$; thus $r \cos \theta = 5$ and $r \cos \theta = -5$, are perpendicular to the initial line, and the latter may be written $r \cos (\theta - 180°) = 5$.

If $a = 90°$, the line is parallel to the initial line, and since $\cos (\theta - 90°) = \cos (90° - \theta) = \sin \theta$, the equation reduces to

$$r \sin \theta = p,$$

in which, if p is positive, the line is above the initial, line, if p is negative, below it; the proper value of a in the latter case being $270°$. Thus $r \sin \theta = 3$, $r \sin \theta = -3$, are parallel to the initial line.

Examples.—Give the polar equation of the line whose nearest point is $(5, 45°)$; that for which $a = 60°, p = 3$; that for which $a = 60°, p = -3$.

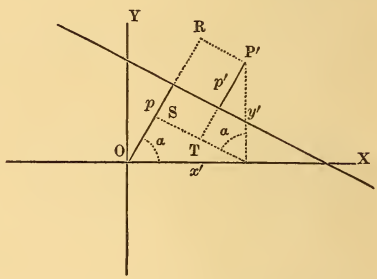
Give the equation of parallels to the initial line, six units above it and below it.

72. Owing to the difficulty of working with trigonometric equations, the polar equations are not well adapted to finding the intersections of lines. The following is one of the simplest class of examples: Find the intersection of $r \cos \theta = 7$ and $r \sin \theta = 5$. We eliminate r by division and obtain $\tan \theta = \frac{5}{7}$, from which, by the trigonometric tables, we find the value of θ to be $35° \, 32' \, 15''$. We might also take $\theta = 215° \, 32' \, 15''$, an angle having the same tangent as $35° \, 32' \, 15''$, but having a negative sine and cosine; but we take θ in the first quadrant, so that r, as found from $r \cos \theta = 7$ or $r \sin \theta = 5$, will be positive. This value will be found to be approximately $r = 8.6023$. If we wish to find only the value of r, we may eliminate θ from the original equations by squaring and adding, for $\cos^2 \theta + \sin^2 \theta = 1$; hence $r^2 = 74$, $r = \pm \sqrt{74}$ according as we take θ in the first or third quadrant. The inter-

section of the lines $n \cos \theta = 7$ and $r \sin \theta = 5$, is the point $(\sqrt{74}, 35° \, 32' \, 15'')$ or $(-\sqrt{74}, 215° \, 32' \, 15'')$.

Distance of a Given Point from a Given Line.

73. We suppose, now, the axes to be rectangular, and that it is required to find the distance of a point given by its co-ordinates, from a line given by its equation. Instead of finding the co-ordinates of the foot of the perpendicular, and using the formula of Art. 12, we may put the equation in the form $x \cos a + y \sin a = p$, in which p is the value of the perpendicular from the origin. Then, evidently, the value of the expression $x' \cos a + y' \sin a$ would be p, if the point P′ were on the line; if not, it is the perpendicular from the origin on a parallel line passing through P′. Thus in the figure, the construction of which is apparent, the value of $x' \cos a + y' \sin a$ is the length of OR. The difference between this and p is the perpendicular required, which denote by p'; then



$$x' \cos a + y' \sin a - p = p'$$

is the formula for the perpendicular from P′ upon the line

$$x \cos a + y \sin a - p = 0.$$

In fact, the formula is equivalent to an equation of condition, expressing that P′ is on the line,

$$x \cos a + y \sin a = p + p',$$

which is the equation of the parallel through P′, of which RP′ is a part.

74. The dotted lines of the figure are drawn to indicate a geometric or direct method of proving the formula; for OR $=$ OS $+$ P′T; now by the definitions of trigonometry OS $= x' \cos a$, and P′T $= y' \sin a$. Therefore $p' =$ OR $- p = x' \cos a + y \sin a - p$.

We may now regard the equation of a line, when in the form $x \cos a + y \sin a - p = 0$, as expressing that the point whose co-ordinates are x and y, is at *no* distance from the line; that is, on the line; and when, on substituting the co-ordinates of any given point for x and y, the equation is not satisfied, the value of the first member expresses the distance of the point from the line, and its sign shows on which side of the line it is situated.

75. In applying this formula, it is necessary first to reduce the given equation to the proper form, by dividing through by the square root of the sum of the squares of the coefficients, as explained in Art. 52. But as we are not concerned with the value of a, it is *not* necessary to attend to the sign of p, as in Art. 54. Thus, given $12y - 5x + 26 = 0$, we divide by 13, and the result is $\frac{12}{13}y - \frac{5}{13}x + 2 = 0$, therefore $\frac{12}{13}y' - \frac{5}{13}x' + 2 = p'$ is the formula for the perpendicular from a given point. If we require the perpendicular from the point (3, 2), substitution of the values gives $p' = 2\frac{9}{13}$; for the point $(1 - 2)$, $p' = -\frac{3}{13}$. The opposite signs of these values show that the points are on opposite sides of the line. If we make $x' = 0$ and $y' = 0$, we shall have the absolute term, which is therefore the perpendicular from the origin; in this case it is two units in length and has the positive sign. As all points on the same side of the line have perpendiculars of the same sign, we find that a point is on the *same side as the origin*, when the *perpendicular has the sign of the absolute term;* thus (3, 2) is on the same side of the line with the origin, but $(1, -2)$ is on the side remote from the origin.

EXAMPLES.—Find the length of the perpendicular from the point (2, 3) on the line $2x + y - 4 = 0$.

Ans. $\dfrac{3}{\sqrt{5}}$, and the point is on the side remote from the origin.

Find the perpendiculars from each vertex to the opposite side of the triangle, whose vertices are $(1, 2)$, $(-2, 0)$ and $(6, -1)$.

Supposing the axes rectangular, what is the general expression for the perpendicular from P′ on any line?

$$\textit{Ans.}\quad \frac{Ax' + By' + C}{\sqrt{A^2 + B^2}}.$$

What is the distance of P′ from the lines $x = 0$ and $y = 0$?

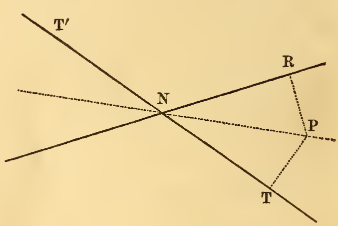
Formula for Line Bisecting the Angles of Given Lines.

76. We have seen that $Ax + By + C + k(A'x + B'y + C') = 0$ represents a line, passing through the intersection of two given lines, k being an arbitrary constant, the value of which we determined in the examples of Art. 60 by equations of condition. If the equations of the given lines be put in the form $x \cos a + y \sin a - p = 0$ (the axes being still supposed rectangular), we have

$$x \cos a + y \sin a - p + k(x \cos a' + y \sin a' - p') = 0.$$

We have now proved that the expression $x \cos a + y \sin a - p$ is the value of the perpendicular from any point upon the line, whose equation is $x \cos a + y \sin a - p = 0$; that is, on the first of the two given lines. The expression in brackets is the perpendicular upon $x \cos a' + y \sin a' - p' = 0$, the other given line. Therefore the equation above asserts that the perpendicular from P, on the first line, equals $-k$ times the perpendicular upon the second. Thus, $12x + 5y + 9 = 0$ and $4x - 3y + 10 = 0$ being the given lines, if we reduce the equations to the proper form and combine them, we shall have $\frac{12}{13}x + \frac{5}{13}y + \frac{9}{13} + k(\frac{4}{5}x - \frac{3}{5}y + 2) = 0$, expressing the condition that the perpendiculars on the two lines shall be in the constant ratio of $-k : 1$.

It is easy to see, in general, that this condition is fulfilled by all points on a certain line, passing through the intersection of the given lines. For let NR and NT be the given lines, and NP any line passing through N, then the ratio of the perpendiculars, PR and PT, from any point of this line is the same as the ratio of the *sines* of the angles PNR, PNT, into which the whole angle RNT is divided.



77. The position of the line PN depends upon this ratio of perpendiculars; that is, on the value of k. If $k = 1$ or $k = -1$, the perpendiculars are equal, and either the angle RNT or the supplementary angle RNT' is bisected; therefore

$$x \cos a + y \sin a - p \pm (x \cos a' + y \sin a' - p') = 0$$

are the equations of the lines bisecting the angles of the given lines. The position of the lines with respect to the axes is immaterial, except as the position of the origin determines the sign of the perpendicular. Thus, in the figure, suppose the origin to be situated within the angle RNT, and the absolute terms to have the same sign; then the perpendiculars PR and PT (which, by Art. 75, will have the signs of the absolute terms) will have the same sign, and k must be negative. That is, the *absolute terms* in the combined equations *having the same sign, k must be negative* for a line passing through *that angle in which the origin is situated.* Thus, in $\frac{12}{13}x + \frac{5}{13}y + \frac{9}{13} + k\left(\frac{4}{5}x - \frac{3}{5}y + 2\right) = 0$, negative values of k give lines passing in the neighborhood of the origin. This criterion is readily remembered by considering the value that k must have in order that the line pass *through* the origin—namely, $k = -\frac{C}{C'}$, as explained in Art. 62.

Now giving k the values $+1$ and -1, in the example, and clearing of fractions, we have $112x - 14y + 175 = 0$ and $8x + 64y - 85 = 0$ for the equations of the two bisectors of the angles of $12x + 5y + 9 = 0$ and $4x - 3y + 10 = 0$. Since these lines bisect supplementary angles, they are mutually perpendicular; accordingly, on reducing them to the form $y = mx + b$ we find the value of m in the first to be 8, and in the other $-\frac{1}{8}$, which is its negative reciprocal, as in Art. 48.

78. In general, the equations of any two lines referred to rectangular axes being $Ax + By + C = 0$ and $A'x + B'y + C' = 0$, reducing them to the proper form and combining, making $k = \pm 1$, and finally clearing of fractions, we have the equations

$$\sqrt{A'^2 + B'^2}\,(Ax + By + C) \pm \sqrt{A^2 + B^2}\,(A'x + B'y + C') = 0$$

for the lines bisecting their angles. If now we take the two values of m from these equations and multiply them together, we shall find their product to be -1, showing that they are of opposite signs, and reciprocal, which is a general proof that the lines are perpendicular.*

* By trigonometric analysis it may be shown that $x \cos a + y \sin a - p \pm (x \cos a' + y \sin a - p') = 0$ have the inclinations $90° + \frac{1}{2}(a + a')$ and $\frac{1}{2}(a + a')$.—*Chauvenet's Plane Trigonometry*, Eqs. 111 and 114.

EXAMPLES.—Find the line bisecting that angle of $3x + 2y + 5 = 0$ and $x + y + 2 = 0$, which contains the origin.

Ans. $(3\sqrt{2} - \sqrt{13})x + (2\sqrt{2} - \sqrt{13})y + 5\sqrt{2} - 2\sqrt{13} = 0$.

Find bisectors of the angles of $2x + y + 8 = 0$ and $x + 2y - 3 = 0$; of the angles of $7x + y = 3$ and $x = y$.

Ans. $12x - 4y - 3 = 0$ and $2x + 6y - 3 = 0$.

Find lines bisecting the angles which $3x - 4y + 15 = 0$ makes with the axis of X ($y = 0$).

Ans. $x - 3y + 5 = 0$ and $3x + y + 15 = 0$.

Find bisectors of the angles of these last lines, and finally bisectors of the results.

Notice that the equations of perpendicular lines when in their simplest forms may be added and subtracted at once, the radicals which reduce the equations to the proper form being the same. The bisectors of the angles of perpendiculars are inclined to them at angles of 45°, and if we repeat the operation on them we return to the original lines.

Prove this to be generally true by means of the equations $y - mx - b = 0$ and $my + x - a = 0$.

EQUATIONS REPRESENTING TWO OR MORE LINES.

79. If from the equations of two loci, a third equation be derived by any algebraic process, its locus will pass through all the points of intersection of the original loci. For the original equations are both true of any one of these points, hence the derived equation must be true also.

For example, $y = 2x + 3$ and $y = x + 1$ represent certain straight lines: multiply them member by member; the result is $y^2 = 2x^2 + 5x + 3$, which represents a curve, of which all we know at present is, that it passes through the intersection of the given lines, which may be easily verified.

We have already examined the effect of adding the equation of straight lines, and found a general equation including all the results which can be derived in that way, introducing an arbitrary multiplier k, because the result is affected by previously multiplying one of the equations through by any number, or both of the equations by different numbers.

80. If we are to multiply two equations member by member, it

is plain that previously multiplying one of them by any number, has no effect upon the locus represented by the result; but previous transposition of a term from one member to the other has an effect upon the result.

We now suppose all the terms transposed to one member in each case, and these members multiplied; for example, the above straight lines would thus give $(y-2x-3)(y-x-1)=0$. Such an equation will be satisfied, not only by the point or points which satisfy *both* the given equations, but by all points which satisfy *either*, and by no other points. For to satisfy the equation, one or other of the factors must become zero; that is, the point must be on one of the given lines.

In general, if $S=0$ and $S'=0$ represent the given equations (S and S' representing polynomials of any degree containing x and y), $SS'=0$ is satisfied by all points, which make either $S=0$, or $S'=0$, that is, by all points of both the given loci. We shall therefore call it the *compound* equation of the given loci, reserving the term *combined* equation for $S+kS'=0$, which is satisfied by those points which make $S=0$ and $S'=0$ *simultaneously*, and also by certain points which make neither $S=0$ nor $S'=0$.

81. The general compound equation of two straight lines is

$$(Ax+By+C)(A'x+B'y+C')=0:$$

when expanded we find it to be of the second degree, thus the compound equation of $y-2x-3=0$ and $y-x-1=0$ is $y^2-3xy+2x^2-4y+5x+3=0$. We see therefore that an equation of second degree *may* represent a pair of straight lines. Given the above equation, we should know its locus completely, if we could resolve the first member into its factors; but an expression of the second degree, containing x and y, is *not generally* resolvable into factors; the condition on which it is resolvable belongs to the discussion of the general equation of second degree.

82. If the two factors are the same, that is, if the expression is a perfect square, the two lines coincide; thus

$$(Ax+By+C)^2=0$$

is satisfied only by the points of a single line; but being an equation of the second degree, its locus is said to be "a pair of coinci-

6

dent lines;" thus $x^2 - 4xy + 4y^2 = 0$ is the equation of two lines coincident with $x - 2y = 0$ or $x = 2y$; $x^2 = 0$, of two lines coincident with the axis of Y.

In some instances the equation can be reduced by the ordinary methods of solution, or the expression in the first member can be factored at sight: thus $x^2 = y^2$ reduces to $x = \pm y$, $x^2 - 3x = -2$, to $x = 1$ or $x = 2$; and $x^2 - y^2 = 0$ is equivalent to $(x + y)(x - y) = 0$, $x^2 - 3x + 2 = 0$, to $(x - 1)(x - 2) = 0$, $xy - 10x = 0$, to $x(y - 10) = 0$.

Examples.—Form the compound equation of $2x - y = 3$ and $y - 4x = 6$; of the parallel lines $y = mx + b$ and $y = mx - b$; of $x = a$ and $y = b$; of the perpendiculars $y - mx - b = 0$ and $my + x - a = 0$.

What is the locus of $xy = 0$? of $x^2 = 4y^2$? of $y^2 = 3y$?

CHAPTER III.

TRANSFORMATION OF CO-ORDINATES.

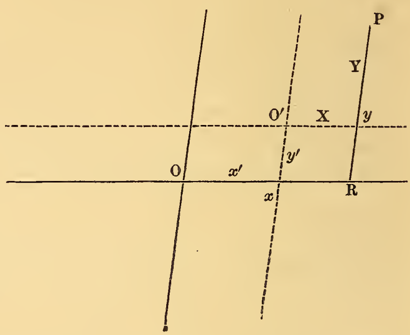
83. HAVING found the equation of a locus as referred to certain lines as axes, it may be desired to find its equation as referred to other axes, or to a new system of co-ordinates. In other words, given a relation between certain co-ordinates of a point, let it be required to find an equivalent relation between other co-ordinates of the same point.

The co-ordinates which occur in the given equation are called the *old* co-ordinates, the others the *new* co-ordinates; if now we find values of the old co-ordinates in terms of the new, and substitute them in the given equation, the result will be an equation between the new co-ordinates, which will be true of every point for which the old equation was true. Hence it will be the required equation; in finding it we are said to *transform* the equation to new co-ordinates.

84. The old co-ordinates or variables of the given equation being denoted by x and y, we shall for distinction use X and Y to denote the new co-ordinates or variables which will appear in the new equation. Certain constants determining the position of the new axes relatively to the old ones must be introduced, which are called *constants of transformation.* To simplify the preparation of formulæ, we consider first the case in which the origin is moved, afterward those in which we pass to the system of polar co-ordinates, or change the direction of the axes.

85. CASE I.—TO PASS TO A SYSTEM OF PARALLEL AXES WITH A NEW ORIGIN.

In the figure let O be the old, and O′ the new origin; let x' and y' be the co-ordinates of O′, relative to the old axes. Take any point, P, and draw PR, or y, its ordinate, to the old axes; it is divided by the new axis of X into two parts, of which one is the new ordinate of P or Y, and the other equals the old ordinate of O′ or y'. Similarly, x consists of two parts, X and x'; hence the formulæ,



$$x = X + x', \quad y = Y + y'.$$

For example, to transform an equation to parallel axes intersecting in the point (4, 3), we have $x = X + 4$ and $y = Y + 3$ for formulæ of transformation. Applying them to the equation of the straight line $x + 2y = 10$, gives $X + 4 + 2Y + 6 = 10$ or $X + 2Y = 0$. This equation is true with respect to new co-ordinates, wherever $x + 2y = 10$ is true of the old co-ordinates. The want of an absolute term shows that the line passes through the new origin, as may easily be verified, and we observe it to have the same direction ratio, as we should expect since the axes are parallel.

EXAMPLES.—Transform $2x + 3y = 6$ to the new origin $(1, -2)$, etc., etc.

Transform $xy + 2x - y = 2$ to the origin $(1, -2)$.

Transform $y - y' = m(x - x')$ to the new origin P′ (x', y').

Transform $x^2 + y^2 = 25$ to the origin $(-4, 3)$, and the result to a third origin whose co-ordinates referred to the second axes are $(7, -7)$.

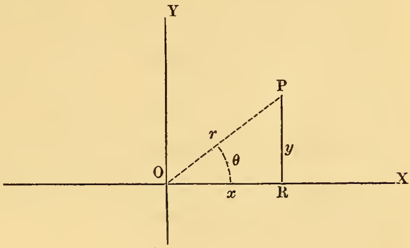
Find the co-ordinates of this third origin as referred to the original axes, and verify by transforming the original equation directly.

86. CASE II.—TO PASS FROM RECTANGULAR TO POLAR CO-ORDINATES.

We here suppose the pole to coincide with the origin, and the initial line with the axis of X. Then drawing the ordinate and radius vector of any point, P, we have the right-angled triangle PRO, in which the angle POR is denoted by θ. From the definitions of trigonometry, $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$; hence

$$x = r \cos \theta, \; y = r \sin \theta.$$

Owing to the simplicity of these formulæ the polar and rectangular equations are usually treated in connection.



In the figure r is positive, because θ is always the angle between the positive directions of r and x; therefore the functions of θ are positive in the first quadrant, and in all the quadrant follow the signs of x and y. If we consider r, in the figure, as negative, θ must be increased by 180°; $\cos \theta$ and $\sin \theta$ will then both be negative, and the formulæ still give positive values for x and y.

EXAMPLES.—Transform from rectangular to polar co-ordinates $x^2 + y^2 = 36$; $2x - y = 10$; $x = y$,

Ans. $r = 0$ and θ indeterminate (the origin), or else $\theta = 45°$ (straight line through the origin); $y = 2x$,

Ans, $r \sin \theta = 2r \cos \theta$ or $\tan \theta = 2$, $\theta = 63° \, 26' \, 6''$.

87. If the pole is to be at any other point than the origin, we must add its co-ordinates, x' and y', to the above values of x and y; or, which is more simple in practice, transform by Case I. to the given point as origin, and then pass to polar co-ordinates. If the initial line does not coincide with the axis of X, let α denote its inclination, then the angle POR will be $\theta + \alpha$ instead of θ. Hence the formulæ become

$$x = r \cos (\theta + \alpha), \; y = r \sin (\theta + \alpha).$$

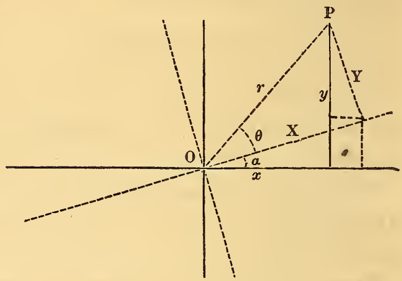
For example, to transform to a system in which the initial line makes an angle of 45° with the axis of X, the formulæ are $x = r \cos (\theta + 45°)$, $y = r \sin (\theta + 45°)$. By the rules of "angular analysis" they may be expanded to $x = \frac{r}{\sqrt{2}} (\cos \theta - \sin \theta)$, $y = \frac{r}{\sqrt{2}} (\cos \theta + \sin \theta)$.—[See *Chauvenet's Plane Trig.*, Eq. 149.] Applying them to the line $x + y = 4$, we find $2 \frac{r}{\sqrt{2}} \cos \theta = 4$, or

$r \cos \theta = 2\sqrt{2}$ for its equation in the required polar system. The line is therefore perpendicular to the new initial line, and at a distance $2\sqrt{2}$ from the pole, by Art. 71.

EXAMPLES.—Transform $x = 7 + y$ and $2x = y$, to the above system.

88. CASE III.—TO PASS FROM ONE RECTANGULAR SYSTEM TO ANOTHER WITH THE SAME ORIGIN.

As rectangular co-ordinates are most frequently employed in the applications of analysis, we separate this case from the general one of change in the direction of axes, and treat it in connection with polar co-ordinates and the angular analysis.



Let a denote the inclination of the new axis of X to the old, and r and θ, the polar co-ordinates of P, referred to it as initial line. Then X and Y denoting the new co-ordinates, we have $X = r \cos \theta$ and $Y = r \sin \theta$; but expanding the values in the last Article, $x = r \cos (\theta + a)$ and $y = r \sin (\theta + a)$, we have

$$x = r \cos \theta \cos a - r \sin \theta \sin a,$$
$$y = r \sin \theta \cos a + r \cos \theta \sin a;$$

substituting X and Y for their values, we have the required formulæ,

$$x = X \cos a - Y \sin a,$$
$$y = Y \cos a + X \sin a.$$

The student familiar with trigonometry may recall these formulæ at any time, by connecting them with the formulæ of expansion, for sine and cosine of the *sum* of two angles; remembering that x corresponds to cosine, being measured horizontally, and y to sine, being measured perpendicularly.

89. By drawing, from the foot of the ordinate Y, lines perpendicular and parallel to the old axis of X, it is easy to see that we construct lines equivalent to the terms of which x is the difference, and

to those of which y is the sum, according to the formulæ. We have thus a direct proof of the formulæ, which is, in fact, precisely the same as that by which the formulæ of trigonometry above referred to was originally proved.*

If the new axis of X is on the other side of the old one, a must be considered negative or in the fourth quadrant; that is, cos a remains positive, but sin a is negative, so that x is the sum of two terms and y the difference of two terms, as in the trigonometric formulæ for cosine and sine of the *difference* of two angles. If $a = 90°$, the formulæ become $x = -Y$, $y = X$, showing that the abscissa has become an ordinate and the reverse, but the negative direction of the new axis of Y corresponds with the positive direction of the old axis of X. Similar interpretations may be given to the cases $a = 180°$, $a = 270°$, $a = 360°$.

EXAMPLES.—Transform $2x - y = 3$ from rectangular axes to others making $a = 45°$; to axes inclined at 60° ($\sin 60° = \frac{1}{2}\sqrt{3}$ $\cos 60° = \frac{1}{2}$).

Transform $3x = 2 - 4y$ by turning the axes back 30° ($a = -30°$), and the result by turning them forward 90°.

Verify by turning the axes of the original equation forward 60°.

Transform $x^2 + y^2 - 4x - 6y = 0$ to the origin (2, 3) and axes bisecting the angles of the given axes.

Transform $x^2 + y^2 = R^2$ by turning the rectangular axes through the angle θ.

Ans. $X^2 + Y^2 = R^2$; the equation is not altered by turning the axes through any angle which can be a property of no line except a circle with its centre at the origin

Transform $x \cos a + y \sin a = p$ in the same manner.

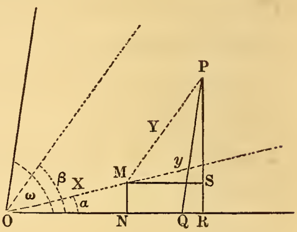
Ans. By angular analysis the equation reduces to $x \cos (a - \theta) + y \sin (a - \theta) = p$, which is still in the form of Art. 51.

90. CASE IV.—TO PASS FROM ANY CARTESIAN SYSTEM TO ANY OTHER WITH SAME ORIGIN.

This is the general case of change of direction; it will be necessary to introduce two constants of transformation—namely, angles determining the new directions of the two axes. Let a denote the inclination of the new axis of X, and β that of Y, to the old axis

* Compare with the above figure, Fig. 10, *Chauvenet's Plane Trig.*

of X; and let ω denote the angle between the old axes. Draw the ordinates of P, and also from P a perpendicular PR, to the old axis of X, and from the foot of the new ordinate, Y, a perpendicular and parallel. By the right-angled triangles PQR, MON and PMS, which contain angles equivalent to ω, α and β, $PR = y \sin \omega$, $SR = X \sin \alpha$, $PS = Y \sin \beta$; hence



$$y \sin \omega = X \sin \alpha + Y \sin \beta.$$

If a perpendicular be drawn from P to the old axis of Y, and a figure constructed in a similar manner (drawing the abscissas of P), we may prove

$$x \sin \omega = X \sin (\omega - \alpha) + Y \sin (\omega - \beta),$$

$\omega - \alpha$ and $\omega - \beta$ being the inclinations of X and Y to this axis, as α and β were to the axis of X.

The formulæ will be most easily remembered, in the above forms which express that "either co-ordinate, into the sine of its inclination to the axis it cuts, equals the sum of the new co-ordinates multiplied each by the sine of its inclination to the same line;" both members of this equality being the true distance of the point from the axis. But the formulæ for substitution are

$$x = X \frac{\sin (\omega - \alpha)}{\sin \omega} + Y \frac{\sin (\omega - \beta)}{\sin \omega},$$

$$y = X \frac{\sin \alpha}{\sin \omega} + Y \frac{\sin \beta}{\sin \omega}.$$

The angles α and β are between the positive directions of the axes; if either of the new axes is on the other side of the old axis of X, the corresponding angle is negative, and its sine, and one term in the formula for y is negative. If one of the new axes is on the other side of the old axis of Y, $\omega - \alpha$ or $\omega - \beta$ becomes negative. If we make $\omega = 90°$, so that the old axes are rectangular, and $\beta = \alpha + 90°$, so that the new are also rectangular, the formulæ reduce to those of Case III. In using these formulæ the values of the four fractions should be first computed, and the formulæ simplified before substituting. Thus, to transform from axes making an angle of $60°$ to axes bisecting their angles: here $\omega = 60°$, $\alpha = 30°$, $\beta = 120°$; hence $x = \frac{X}{\sqrt{3}} - Y$ and $y = \frac{X}{\sqrt{3}} + Y$.

EXAMPLES.—Transform by these formulæ $x^2 + y^2 = 100$; $y = x + 3$; $x + y = 6$.

Transform the same lines to axes making $\alpha = 45°$, $\beta = 90°$, $(\omega = 60°)$.

TRANSFORMATION OF A POINT.

91. The formulæ we have given for transformation are values of the *old* co-ordinates in terms of the *new*, because their principal application is to the transformation of *equations*. If we have the co-ordinates of a point in one system, and require its co-ordinates in a new system, it would be more convenient to have formulæ, giving the value of X and Y, in terms of x and y. Such formulæ we may call *reverse* formulæ of transformation. Thus, to pass from a rectangular system to another inclined to it by $60°$, the formulæ of Case III. become

$$x = \tfrac{1}{2} X - \tfrac{1}{2}\sqrt{3} \,.\, Y, \qquad y = \tfrac{1}{2} Y + \tfrac{1}{2}\sqrt{3} \,.\, X.$$

Solving these equations for X and Y, we have

$$X = \tfrac{1}{2} x + \tfrac{1}{2}\sqrt{3} \,.\, y, \qquad Y = \tfrac{1}{2} y - \tfrac{1}{2}\sqrt{3} \,.\, x.$$

If then we wish to transform the point (4, 2), we find by substitution its new co-ordinates $X = 2 + \sqrt{3}$, $Y = 1 - 2\sqrt{3}$.

92. These reverse formulæ might be found also by interchanging x and X, y and Y, in the original formulæ, and making the proper changes in the constants. Thus in Case I. they are $X = x - x'$, $Y = y - y'$, in which the constants are $-x'$ and $-y'$, because these are the co-ordinates of the *old* origin referred to the *new* axes. In Case III. α becomes $-\alpha$, because we reverse the direction in which we turn the axes.

EXAMPLES.—Transform (3, 3) from rectangular axes to axes bisecting their angles; the point $(1, -2)$ to the same axes.

Transform $(\sqrt{3}, 1)$ by turning the axes three times in succession through $120°$. The final result should be $(\sqrt{3}, 1)$.

Verify the reverse formulæ of Case III., by finding X and Y by elimination.

93. The equations of a point are in reality the equations of two lines of the forms $x = a$, $y = b$, parallel to the old axes, and intersecting in the point. Substituting for x and y their values by the formulæ of transformation, we have the *new equations* of these two straight lines, from which by elimination we can find the new co-ordinates of the point of intersection. The process is the same as when we find X and Y in terms of x and y, by combining the formulæ; the known co-ordinates a and b merely taking the place of the general co-ordinates x and y. Thus, in the example of Art. 91, the formulæ are $x = \frac{1}{2}\mathrm{X} - \frac{1}{2}\sqrt{3}.\mathrm{Y}$, $y = \frac{1}{2}\mathrm{Y} + \frac{1}{2}\sqrt{3}.\mathrm{X}$; hence

$$\tfrac{1}{2}\mathrm{X} - \tfrac{1}{2}\sqrt{3}.\mathrm{Y} = a \qquad \text{and} \qquad \tfrac{1}{2}\mathrm{Y} + \tfrac{1}{2}\sqrt{3}.\mathrm{X} = b$$

are the new equations of the lines which determine the point (a, b). Their intersection is the point $\mathrm{X} = \frac{1}{2}a + \frac{1}{2}\sqrt{3}.b$, $\mathrm{Y} = \frac{1}{2}b - \frac{1}{2}\sqrt{3}.a$.

94. If we put $a = 0$ and $b = 0$, we have the *new equations of the old axes*, which would lead us to expect that the formulæ would all be, as we have found them, *linear*, or of the first degree with respect to X and Y; also, that in those cases where the origin is not moved, there would be no absolute term. Thus in Case III., if $a = 45$, the formula for x, is $x = \frac{1}{2}\sqrt{2}\,(\mathrm{X} - \mathrm{Y})$ and $\frac{1}{2}\sqrt{2}\,(\mathrm{X} - \mathrm{Y}) = 0$, or $\mathrm{X} = \mathrm{Y}$, is the new equation of the old axis of Y, or line whose old equation was $x = 0$.

Transformation of Formulæ.

95. The method of transformation may be applied to formulæ for lines in particular positions, so as to produce more general formulæ for the same lines. Suppose, for instance, we know the form of the equation of a certain line, when referred to axes passing through a particular point, and we require its equation when this point has any position P′, the former lines of reference being parallel to the axes. Thus, the equation of a circle referred to any rectangular axes passing through the centre is $x^2 + y^2 = \mathrm{R}^2$, what is the equation when the centre is situated at P′?

Since the co-ordinates of the centre, x', y', are given, we may regard that point as the new origin of Case I.; and the given equation must be written $X^2 + Y^2 = R^2$, because it is a relation between co-ordinates, as measured from that point. We thus have the *new* equation to transform *back* to the *old* axes by the reverse formulæ,

$$X = x - x', \qquad Y = y - y',$$

which gives $(x - x')^2 + (y - y')^2 = R^2$, for the circle whose centre is P′. Thus, having the equation of a line, *a particular point connected with it being at the origin*, we find a more general formula by substituting $x - x'$ and $y - y'$ for x and y; the new constants x' and y' being the co-ordinates of the particular point.

We have already met with an instance of this in the equation of the straight line, which is $y = mx$, a point of the line being at the origin, and $y - y' = m(x - x')$, a point of the line being at P′.

96. In the formulæ derived on this principle, P′ will not generally be a point *on* the line; but, as above, in the case of the circle, the point to which, as origin, it is most readily referred. We shall frequently use X and Y for co-ordinates measured from this point or *central co-ordinates*, regarding them as abridged symbols for the differences of co-ordinates of point and centre, $x - x'$ and $y - y'$, to which they are equivalent. See Art. 101.

97. The formulæ of Cases III. and IV. might be used in a similar manner when the direction of the axes is changed, but not so conveniently, because the constants introduced would be angles. By Case II. general polar equations may be found from the general rectangular equations; thus, $Ax + By + C = 0$ gives $r(A \cos\theta + B \sin\theta) + C = 0$. Also, when a polar equation can be put into a form in which $r \cos\theta$ and $r \sin\theta$ occur, it can be changed to a rectangular equation; thus, $r \cos(\theta - a) = p$ expanded is $r \cos\theta \cos a + r \sin\theta \sin a = p$, or $x \cos a + y \sin a = p$. In equations of the second degree an additional formula for this reverse transformation,

$$r^2 = x^2 + y^2,$$

will be useful. Thus, $r = 5$, or $r^2 = 25$, becomes $x^2 + y^2 = 25$.

Arbitrary Transformation.

98. The equation of a line or curve depends partly upon its shape, and partly upon its position with respect to the axes. The

circumferences of circles with different radii, for instance, are of different shapes, and hence their equations differ *essentially;* but the equations of the *same* circle in different positions are such that one may be derived from another by transformation. So that given the equation of a line, it is possible that with other axes it might have a simpler equation, from which its shape might be more easily deduced.

To find these simpler equations, we use the formulæ of transformation in their algebraic forms, considering the constants as *arbitrary*, and then examine the transformed equation, to see in what manner it can be simplified by particular determinations of the constants.

99. This general transformation is of two kinds: first, change of origin. For example, $2x - 3y - 4 = 0$, by substituting the formulæ of Case I., becomes $2X - 3Y + 2x' - 3y' - 4 = 0$, in which we can make the absolute term disappear by giving x' and y' such values that $2x' - 3y' - 4 = 0$. Thus, by making the new origin satisfy the original equation, the equation of the line takes the simpler form $2X - 3Y = 0$, which shows that the line passes through the new origin. The second kind is change in direction of axes, by the formulæ of Case IV. For example, by this transformation $y = 3x$ becomes $X\left(\frac{\sin \alpha}{\sin \omega} - 3\frac{\sin(\omega - \alpha)}{\sin \omega}\right) + Y\left(\frac{\sin \beta}{\sin \omega} - 3\frac{\sin(\omega - \beta)}{\sin \omega}\right) = 0.$ If we give α such a value as to make the coefficient of X equal zero, the equation will reduce to $Y = 0$; that is, the line will coincide with the new axis of X. The condition is $\frac{\sin \alpha}{\sin(\omega - \alpha)} = 3$, which is satisfied when α equals the angle of the line's inclination. (See note to Art. 48.)

EXAMPLES.—To what origin must we transform $x^2 + y^2 - 4x + 2y - 4 = 0$ to make it take the form $X^2 + Y^2 = R^2$? to what, in order to make it have no absolute term?

The axes being rectangular, through what angle must we turn them to make $4x = y + 1$ take the form $Y = b$? (by Case III.)

100. This method may be used to simplify general equations. The general equation of first degree, $Ax + By + C = 0$, may by change of origin be freed from its absolute term, and then, by change of direction of axes, be reduced to the form $Y = 0$, which repre-

sents the axis of X; that is, the axis can always be made to coincide with $Ax + By + C = 0$. The lines represented by this equation differ only in position, being in fact all *straight* lines; but the lines represented by general equations of higher degrees differ not merely in position, but essentially in shape.

The constants of transformation are four in number, x' and y', which are introduced by the first transformation, and the angles α and β by the other (ω is not arbitrary, as it depends on the *old* axes). These constants are to be determined by the conditions that the coefficients of certain terms in the transformed equation shall be zero, so that those terms may drop out of the equation. The degree of the equation, however, cannot be altered; for, the values of x and y in the formulæ being of the first degree with respect to X and Y, transformation cannot introduce terms of a higher degree, or *raise* the degree; neither can it *lower* the degree, because transformation *back*, which should restore the original equation, cannot raise the degree. Hence the degree of the equation is fixed, and naturally becomes a basis for the classification of lines. The formulæ of Case II. show that the degree of a polar equation with respect to r is the same as that of the rectangular, and hence of any Cartesian equation of the same line.

It may be shown that the absolute term, for any origin, P′, is the result of substituting x' and y' for x and y, in the first member of an equation, whose second member is zero. For suppose Ax^m to be a term; in the transformed equation, we have $A(X + x')^m$, expanding this by the binomial theorem, the last term Ax'^m is part of the new absolute term. In like manner, each of the transformed terms contains an absolute part; and the new absolute term consists of all these parts, together with the old absolute term. Compare the example in the last Article.

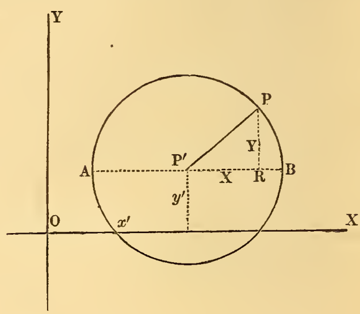
CHAPTER IV.

THE CIRCLE.

101. In treating the circle we shall use rectangular axes only, because the rectangular equation of this curve is the most simple and best adapted to investigating its properties. Its oblique equation will afterward be given, in connection with the general equation of a class of curves of which the circle will be found to be a special case.

If the centre of the circle be at the origin, the equation is $x^2 + y^2 = R^2$, where R is the radius, and it was shown, in Art. 95, that consequently the general equation, when x' and y' denote the co-ordinates of the centre, is

$$(x - x')^2 + (y - y')^2 = R^2.$$



To make the meaning of the equation perfectly clear, draw a circle from any point P′ as centre, and with a radius whose length is denoted by R. Let the co-ordinates of any point of the circumference, as P, be x and y; and let the lines P′R and PR, which are the co-ordinates of P as measured from the centre, or *central co-ordinates of* P, be denoted by X and Y. Then by the right-angled triangles PRP′, $P'R^2 + PR^2 = P'P^2$; hence

$$X^2 + Y^2 = R^2$$

is the *central equation;* and since, in general co-ordinates, P′R is denoted by the difference $x - x'$, and PR by $y - y'$, the above is the general equation. If P be taken in a different part of the circumference, the differences $x - x'$ and $y - y'$ may become negative, but their *squares* will still be positive and equivalent to $P'R^2$ and PR^2; hence the equation is true for all points of the circumference. The centre P' may be so situated that one of its coordinates is negative, in which case $x - x'$ or $y - y'$ will become the sum of the variable and constant, as in the equation of the circle whose centre is $(-2, 1)$, and whose radius is 3; namely, $(x + 2)^2 + (y - 1)^2 = 9$.

EXAMPLES.—Form the equation of the circle whose centre is the point $(4, -3)$, and radius 5; centre at $(3, 0)$, and radius 2; etc., etc.

102. In the equation of a circle, as indeed in all equations of the second degree, each value of x has corresponding to it two values of y, since substituting a given value for x gives a quadratic equation to find the value of y. For instance, in the circle $(x - 4)^2 + (y - 2)^2 = 9$, let $x = 2$, the result is $(y - 2)^2 = 5$, hence $y = 2 \pm \sqrt{5}$. Either of these values of y, in connection with $x = 2$, will satisfy the equation, as is easily verified. We thus find two points of the line having the same abscissa; $x = 3$, $x = 4$, etc., give likewise two values of y. But $x = 7$ gives $(y - 2)^2 = 0$ or $y = 2$, a single value, and $x = 8$, gives $y = 2 \pm \sqrt{-7}$, which are imaginary values of y; hence there is but *one* point having the abscissa 7, and *no points* having the abscissa 8.

103. In general, solving the equation for y, we have

$$y = y' \pm \sqrt{R^2 - (x - x')^2},$$

which is the equation of the circle, expressing y in terms of x, or making it an *explicit* function. On account of the double sign, we say there are *two* values of y corresponding to *each* value of x; but when $(x - x')^2$ is greater than R^2, these values are *imaginary*, and when $(x - x')^2 = R^2$, they are *equal*. Imaginary values will take place when the quantity under the radical is negative, or $x - x'$ is *numerically* greater than R; and equal values, when $x = x' \pm R$. These latter values are, therefore *limiting* values of the abscissa of P, corresponding in the figure to the points A and B. Thus, in the example of the last Article, the limiting values of x are one, and seven units. There are no points of the curve, on the *left* of the point (1, 2) or on the *right* of (7, 2), but if the abscissa be assumed *between* these values, two points of the curve will be found.

In like manner, two values of x correspond to an assumed value of y, and the limiting values of y are those which make the radical part of the value of x equal zero—namely, $y = y' \pm R$; these values belonging to the highest and lowest points, for each of which $x = x'$. We find, therefore, that an equation of the second degree may represent a line limited in all directions, unlike the equation of first degree, which represents an unlimited line.

104. If the centre is at the origin, $y = \pm\sqrt{R^2 - x^2}$, or the two values of y, corresponding to a given value of x, are equal with contrary signs. The two points having the same abscissa are situated at equal distances above and below the axis of X; the curve is therefore said to be *symmetrical* to this axis. $x^2 + y^2 = R^2$ is also symmetrical to the axis of Y.

If the centre is on the axis of X, and at a distance from the origin equal the radius, the equation becomes $(x - R)^2 + y^2 = R^2$, which is readily shown to be symmetrical to the axis of X, but not to that of Y. In fact any equation containing y^2, and *not* containing y in the *first power*, represents a curve symmetrical to the axis of X, for it is satisfied by equal positive and negative values of y.

Various Equations of the Circle.

105. The formula, which we have found for the circle, contains as constants, the radius and co-ordinates of the centre, and enables us to form the equation when these constants are given. If the centre and a point of the circumference are given, x' and y' are known, and R^2 may be determined by an equation of condition. For example, if the centre is to be the point (3, — 1) the formula gives $(x - 3)^2 + (y + 1)^2 = R^2$; and if (2, 1) is to be a point of the curve, the equation of this condition is found by substituting these last values for x and y, which gives $5 = R^2$, hence $(x - 3)^2 + (y + 1)^2 = 5$ is the required equation. In general, P′ being the centre and P″ the point of the curve, the equation of condition is

$$(x'' - x')^2 + (y'' - y')^2 = R^2.$$

This is the same thing as finding R, or the distance between P″ and P′, by the formula of Art. 12; and the equation of the circle itself may be regarded as expressing that the distance between P and the fixed point P′, shall be constant and equal to R.

EXAMPLES.—Give the equations of the circle having (1, 2) as centre and (3, — 2) in the circumference; (3, — 2) as centre and (1, 2) in circumference; (— 2, 5) as centre and passing through (1, 1); etc., etc.

106. When the properties of the curve are investigated, the algebraic work is simplified by using the central equation $X^2 + Y^2 = R^2$; while at the same time the results are applicable to any circle if we consider X and Y as standing throughout for $x - x'$ and $y - y'$. Thus putting the central equation in the form

$$Y^2 = R^2 - X^2 = (R + X)(R - X),$$

and observing that $R + X$ represents the line AR, and $R - X$ the line BR in the figure, we have the property of any circle, that "the square of the perpendicular, dropped from any point of the circumference on a diameter equals the product of the segments into which it divides the diameter."

107. Expanding the equation in terms of centre and radius, we have

$$x^2 + y^2 - 2x'x - 2y'y + x'^2 + y'^2 - R^2 = 0,$$

an equation containing x^2 and y^2 followed by terms of the first degree, and a constant part which may be represented by a single letter, or absolute term. Any equation of the form,

$$Ax^2 + Ay^2 + Dx + Ey + F = 0,^*$$

may be reduced, by dividing by A, to the form,

$$x^2 + y^2 + dx + ey + f = 0,$$

in which d, e and f stand for the ratios $\frac{D}{A}$, etc. The first of these is the *general* equation of the circle, in which the position of the line depends upon the *ratios* of the constants; the second a form in which it depends upon the *values* of the constants. To construct a circle from its equation in this form, we must find the centre and

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* This form has been chosen, so that the letters shall stand in the same places as in the general equation of the second degree.

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radius. Comparing it with the expanded form above, we see that the coefficient d takes the place of $-2x'$, $e = -2y'$ and $f = x'^2 + y'^2 - R^2$, hence

$$x' = -\tfrac{1}{2}d, \qquad y' = -\tfrac{1}{2}e, \qquad R^2 = x'^2 + y'^2 - f.$$

Thus, given the equation $2x^2 + 2y^2 - 4x + 12y - 32 = 0$, which is in the general form, divide through by 2, and we have $x^2 + y^2 - 2x + 6y - 16 = 0$, in which $d = -2$, $e = 6$ and $f = -16$. Hence, $x' = 1$, $y' = -3$ and $R^2 = 26$; and the equation can be written $(x-1)^2 + (y+3)^2 = 26$, the centre being the point $(1, -3)$ and the radius $\sqrt{26}$. Whatever the values of x' and y', the centre is readily constructed; but if the value of R^2, as found, is negative, R is imaginary, and no circle can be constructed.

EXAMPLES.—Reduce $3x^2 + 3y^2 + 9x - 2y = 0$, and $-x^2 - y^2 + 2x + 4y - 10 = 0$, to the first form of the equation of circle and construct if possible.

108. We see then, that the general equation of the circle includes certain equations, which cannot be constructed in this manner, because the expression for the radius is imaginary. These equations are said to be the equations of *imaginary circles*, and are not satisfied by any points. For such an equation may be reduced to the form $(x-x')^2 + (y-y')^2 =$ *a negative quantity*, which can be satisfied by no values of x and y, because each of the squares in the first member is essentially positive, and their sum cannot be negative. If the equation takes the form

$$(x-x')^2 + (y-y')^2 = 0,$$

it is satisfied only by $x = x'$, $y = y'$, which makes each of the squares zero. We have here an equation satisfied only by a single point, but being of the form of the circle, and the radius taking the value zero, it is said to be the equation of an *infinitesimal circle.*

109. Suppose now, the absolute term f to vary, the other constants in the equation remaining fixed. Since x' and y' depend only upon d and e, the centre of the circle remains in the same position, the radius only changing. We have, therefore, the equations of a series of *concentric* circles. Now the radius (putting $-\tfrac{1}{2}d$ and $-\tfrac{1}{2}e$ in place of x' and y'), is

$$R = \sqrt{\tfrac{1}{4}(d^2 + e^2) - f}.$$

If f is negative or zero, R is *real*, since $\frac{1}{4}(d^2 + e^2)$ is essentially positive. But if f becomes positive and increases, R decreases, until $f = \frac{1}{4}(d^2 + e^2)$, when $R = 0$; and if f increases beyond this value, R becomes imaginary.

110. If we wish to find the equation of the circle passing through given points, we must assume the form

$$x^2 + y^2 + dx + ey + f = 0,$$

in which the values of the constants determine the line. As there are *three* arbitrary constants, three equations of condition may be satisfied, or the circle may be made to pass through *three* given points. The number of arbitrary constants is always the same as the number of points through which the curve may be made to pass, but the general equation contains always one more constant. See Art. 56.

For instance, let a circle be required to pass through (2, 1) (— 1, 3) and (1, — 1), the equations of condition are

$$5 + 2d + e + f = 0,$$
$$10 - d + 3e + f = 0,$$
and
$$2 + d - e + f = 0.$$

Eliminating f, by subtracting the second and third from the first, we have

$$-5 + 3d - 2e = 0,$$
and
$$3 + d + 2e = 0,$$

from which $d = \frac{1}{2}$, $e = -1\frac{3}{4}$; and substituting in one of the first set of equations, $f = -4\frac{1}{4}$. With these values, the assumed equation becomes

$$x^2 + y^2 + \tfrac{1}{2}x - 1\tfrac{3}{4}y - 4\tfrac{1}{4} = 0, \text{ or}$$
$$4x^2 + 4y^2 + 2x - 7y = 17,$$

which may be verified for each of the given points.

111. As the co-ordinates of the centre depend upon d and e, each of the two equations between d and e is equivalent to a relation between the co-ordinates of that point. Put $-2x$ for d and $-2y$ for e, and we have

$$-5 - 6x + 4y = 0,$$
and
$$3 - 2x - 4y = 0,$$

in which x and y refer to the required centre. The first (being derived from the first and second equations of condition) is the condition imposed on the *centre*, or the *locus of the centre* of a circle passing through the first two points; namely, (2, 1) and (— 1, 3). In like manner, the other is the locus of the centre of the circle passing through the first and third points (2, 1) and (1, — 1). The point in which they meet $(-\frac{1}{4}, \frac{7}{8})$ is the required centre.

As the equations are of the first degree, the loci are straight lines, and they may be shown to be perpendiculars passing through the middle points of the lines joining the given points. (Compare Art. 15.) If the lines were parallel, which would be the case if the three points were in one straight line, the centre could not be found; that is, a circle cannot be made to pass through three points in one straight line.

EXAMPLES.—Find the circle passing through (3, 3) (2, —1) and (0, 2); through (1, 1) (0, 2) and the origin.

Find the locus of the centre of a circle passing through (1, 6) and (— 3, 2).

Show generally that the locus of the centre, for two given points, bisects perpendicularly the line joining the points. (P_1 and P_2 being the given points, show the equation of this perpendicular to be equivalent to the equation found as above from general equations of conditions.)

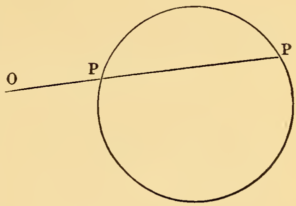
POLAR EQUATIONS OF THE CIRCLE.

112. Using the formulæ of transformation $x = r\cos\theta$, $y = r\sin\theta$, in $x^2 + y^2 = R^2$, we have $r^2 = R^2$ or $r = \pm R$. This is the polar equation when the centre is at the pole. It expresses simply that the radius vector is constant and may be considered either positive or negative; thus $r = 5$ and $r = -5$ denote the same circle. By the same transformation,

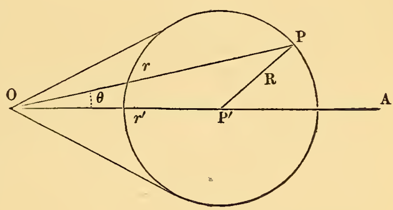
$$r^2 + (d\cos\theta + e\sin\theta)\,r + f = 0$$

is the polar equation of a circle in any position.

This equation being of the second degree, gives two values of r for each value of θ. It is shown in algebra, of quadratic equations in this form (the second member being zero, and the coefficient of the square being unity), that the absolute term is the product of the roots. Therefore the product of the distances from any point, O, to the points P, P, in which a straight line passing through O cuts a circle, is constant and equal to the absolute term of the equation, that point being the origin or pole. If, as in the figure, O is outside of the circle, the two values of r, being a secant and its external segment, are measured in the same direction; therefore their product is positive, or f is positive. If the pole is on the curve, one of the values of r, and consequently their product is zero, and $f = 0$. If the pole is within the circle, the two values of r, being the segments of a chord, are measured in different directions, and f is negative. When the pole is at the centre, $d = 0$ and $e = 0$, the positive and negative values of r become equal to radius, and $f = -R^2$; hence $r = \pm R$.



113. A more useful form of the polar equation may be found, thus: Let r' represent the distance of the centre from the pole; and let the initial line pass through the centre. The rectangular equation is



$$(x - r')^2 + y^2 = R^2,$$

the centre being at the point $(r', 0)$. Expanding and transforming, we have

$$r^2 - 2rr' \cos \theta + r'^2 = R^2.$$

The three lines r, r' and R form the sides of a triangle of which θ is an angle. Hence the equation expresses that "the square of one side of a triangle equals the sum of the squares of the other two sides, minus twice their product, into the cosine of the included angle."

If the centre is not on the initial line, let θ' denote the inclination of r', so that r' and θ' are the polar co-ordinates of the centre. Then the angle POP$'$ is denoted by $\theta - \theta'$, and

$$r^2 - 2rr' \cos(\theta - \theta') + r'^2 = R^2$$

is the polar equation of any circle, in terms of the co-ordinates of its centre and its radius.

EXAMPLES.—Give the polar equation of the circle whose centre is (45°, 7) and radius 10; centre (90°, 5) and radius 2; etc., etc.

114. To express the value of r directly in terms of θ, we complete the square in the first member, by subtracting $r'^2 \sin^2\theta$ (using the equation where $\theta' = 0$). Thus

$$r^2 - 2rr' \cos\theta + r'^2 \cos^2\theta = R^2 - r'^2 \sin^2\theta$$
$$r = r' \cos\theta \pm \sqrt{R^2 - r'^2 \sin^2\theta}.$$

If, as in the figure, $r' > R$, these values of r will become imaginary for some values of θ. For, if $R^2 = r'^2 \sin^2\theta$; that is, if $\sin\theta = \pm\frac{R}{r'}$, the radical part vanishes, and both values of r equal $r' \cos\theta$. These values of θ are only possible when $r' > R$, for the sine of an angle is always a proper fraction. When they are possible, they are limiting values of θ, one in the first and one in the fourth quadrant, and the equal values of r which correspond to them are tangents to the circle. For if $\sin\theta > \frac{R}{r'}$, the values of r are imaginary.

To find the length of the tangent from O, or the equal values of r, we must find what $r' \cos\theta$ becomes, for these values of θ. Now if $r'^2 \sin^2\theta = R^2$, $r'^2 \cos^2\theta = r'^2 - R^2$, hence the length of the tangent is $\sqrt{r'^2 - R^2}$. From this value, or from the value of $\sin\theta$, it is easily shown that a tangent is perpendicular to the radius at its point of contact.

Again, multiplying the two values of r for *any* value of θ, we have $r'^2 \cos^2\theta - (R^2 - r'^2 \sin^2\theta) = r'^2 - R^2$. Or the product of the two values of r, is constant, as shown in Art. 112, and is equal to the square of the tangent. This product is the absolute term of the equation; f, in the previous form, $r'^2 - R^2$, in the present, or $x'^2 - y'^2 - R^2$, in the first rectangular form. When the origin or pole is within the circle, the absolute term is negative and the tangent is impossible.

EXAMPLES.—Reduce $r^2 - 8r \cos \theta = 9$ to the form $r^2 - 2rr' \cos \theta + r'^2 = R^2$, and determine the length of the tangent from the pole, and the limiting values of θ.

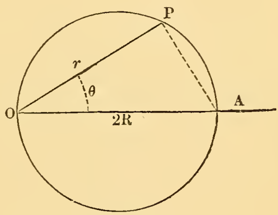
In $-x^2 - y^2 + 2x + 4y - 10 = 0$. $4x^2 + 4y^2 + 2x - 7y = 17$, etc., is the origin within or without the circle? and if without, what is the tangent?

115. If $r' = R$, the polar equation becomes

$$r^2 - 2rR \cos \theta = 0,$$

which is satisfied by $r = 0$, whatever the value of θ; because the pole is on the circumference, therefore *one* of the values of r vanishes or becomes zero. Dividing by r, we have

$$r = 2R \cos \theta,$$



which gives the *other* value of r, and hence is the polar equation of a circle referred to a point on the circumference and a diameter. Or the two values of r may be found by making the supposition $r' = R$ in the two values of r, Art. 114, which gives $r = R \cos \theta \pm R \cos \theta$.

It may be remarked in general of polar equations of the second degree, that if the pole be placed on the line, the absolute term disappears and the equation may be divided through by r. The result is an equation of the first degree for r; but we must remember that in dividing though by r we neglect one root of the quadratic equation—namely, $r = 0$.

If, in $r = 2R \cos \theta$, $\theta = 0$, $r = 2R$, the diameter OA, measured along the initial line. This is the greatest possible value of r, and since any other value of r, as OP in the figure, equals this diameter into the cosine of the included angle θ, the triangle OPA is right-angled at P; or "the angle inscribed in a semi-circle is a right angle." If $\theta = 90°$, this value of r is also zero, or the line perpendicular to OA at O is a tangent. If θ is in the second or third quadrant, r is negative, or the line must be produced backward through the pole to meet the curve.

Intersection of Circle and Straight Line.

116. We saw, in Art. 39, that the intersection of the loci of equations of the first and second degrees may be in two points. The example in that Article was the intersection of the circle $x^2 + y^2 = 25$, with the straight line $25 + x = 7y$, in the two points (3, 4) and (-4, 3). In Art. 40 the same circle is shown *not* to intersect with the line $x = y - 8$, by the occurrence of imaginary quantities in the simultaneous solution of the equations. A circle will therefore generally meet a line in two points, if it meet it at all, and it cannot meet it in more than two points. The same remark would apply to the intersection of the locus of any equation of the second degree, by a straight line.

Tangency to a curve of second degree is indicated by the occurrence of *equal roots;* because then, the line meets the curve in but *one* point. It is therefore a special case, interposed between the more general cases of intersection in two points, and non-intersection. This is exemplified in Art. 102, where we found values of y corresponding to given values of x in the circle $(x - 4)^2 + (y - 2)^2 = 9$, which is the same thing as finding the intersections of the circle with the lines $x = 2$, $x = 3$, etc. $x = 7$ was there the tangent; $x = 6$ would give points of intersection, and $x = 8$ imaginary values.

117. In combining equations of the first and second degree, we should substitute the value of one variable from the first into the second, which gives a quadratic equation. Solving this, we obtain two values of one co-ordinate, each of which must be substituted in the equation of the *first* degree, to find the corresponding value of the other co-ordinate. The work should then be verified by substituting in the equation of the second degree, to see if the points found satisfy that equation. Thus, given the circle $x^2 + y^2 - 2x + y + 1 = 0$, and the straight line $x - y = 1$. Substituting for x its value $y + 1$, we have the quadratic $2y^2 + y = 0$. This is satisfied by $y = 0$ and $y = -\frac{1}{2}$. The corresponding values of x, from $x = y + 1$, are 1 and $\frac{1}{2}$. Therefore the points are (1, 0) and ($\frac{1}{2}$, $-\frac{1}{2}$), which will both be found to satisfy the equation of the circle.

As there are two solutions, which may be verified even when the values are imaginary, a line is said to meet a circle in two *real*, or *coincident*, or *imaginary* points, according as it is a secant, or tangent, or fails to meet the curve.

EXAMPLES.—Find the intersections of $x^2 + y^2 + 6x - 3y = 10$, with the line $x + y = 2$; with the line $y = 3x - 4$; with $x = 4$; etc., etc.

118. In finding general values of the co-ordinates of intersection, we shall for convenience use the central equation,

$$X^2 + Y^2 = R^2,$$

in which X and Y stand for $x - x'$ and $y - y'$. For the straight line we shall use the rectangular form,

$$X \cos a + Y \sin a = p,$$

in which, as X and Y are measured from the centre, p is the perpendicular from the centre. To eliminate Y between these equations, multiply the first by $\sin^2 a$, and substitute for $Y^2 \sin^2 a$ its value from the second; the result is

$$X^2 \sin^2 a + (p - X \cos a)^2 = R^2 \sin^2 a,$$

reducing to $\quad X^2 - 2pX \cos a = R^2 \sin^2 a - p^2.$

Completing square by adding $p^2 \cos^2 a$, we have

$$(X - p \cos a)^2 = (R^2 - p^2) \sin^2 a,$$

since $p^2 = p^2 \sin^2 a + p^2 \cos^2 a$. Hence the values of X,

$$X = p \cos a \pm \sqrt{(R^2 - p^2)} \sin a.$$

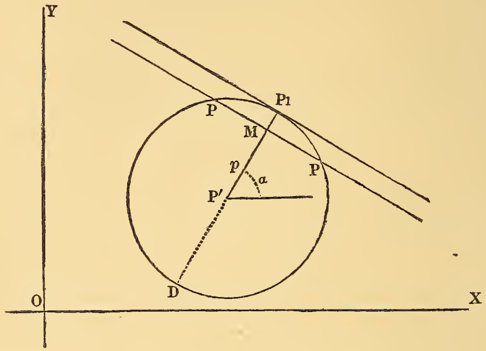
Substituting these values of X, in $Y \sin a = p - X \cos a$, we deduce

$$Y = p \sin a \mp \sqrt{(R^2 - p^2)} \cos a,$$

for the corresponding values of Y, the upper signs being taken together for one point, and the lower signs for the other. The result is verified by squaring the associated values of X and Y, and adding; the sum is found in each case to be R^2.

119. These are the general values for the central co-ordinates of the points P, P, in which the straight line cuts the circle. It is easy to see that the rational parts, $p \cos a$ and $p \sin a$ must be the

co-ordinates of the middle point of the chord PP. But $p \cos a$ and $p \sin a$ are the co-ordinates of M, the foot of the perpendicular, p. A perpendicular from the centre therefore bisects a chord. In



general, the *rational* part of the two values of each co-ordinate of intersection, being an arithmetical mean between them, is a co-ordinate of the middle point of the chord. See Art. 11.

The real or imaginary character of the *radical* part shows whether the line cuts the curve or not. The supposition which makes the radical part zero is intermediate between those which make it positive and negative, and answers to the case in which P, P and M coincide, and the line touching the curve in a single point is called a *tangent*. This method of finding *a condition of tangency*—namely, to place the radical part equal zero—we shall apply to all curves whose equations are of second degree. In this case the condition is that $p^2 = R^2$; a less value of p^2 will make the points P, P, real; a greater value will make them imaginary.

The condition of tangency will always be a *relation* between the *constants* occurring in the equations of the straight line and curve. Its simplicity will of course depend upon the form of the equations employed. We used the central equation for the curve, so as to avoid introducing more than one constant relating to the circle. It happens that the constant a, from the line, does not appear in the condition, so that whatever the direction of the line it will touch the circle, provided only, its distance from the centre has the proper value.

Tangent to the Circle.

120. The condition of tangency, for the circle, being $p^2 = R^2$, or $p = \pm R$, substituting this value of p in the equation of the line gives the equation of a tangent,

$$X \cos a + Y \sin a = \pm R.$$

This line will touch the curve, whatever the value of a, and the double value of p indicates that parallel tangents may be drawn through the two extremities of a diameter, DP_1. The form in which $p = + R$, which, with the value of a in the figure, represents the tangent at P_1, need only be considered, when all values of a from 0° to 360° are admitted; for the tangent at D may be considered as having a value of a, 180° greater than that at P_1.

Putting $p = R$, in the values of X and Y, we find the co-ordinates of P_1,

$$X_1 = R \cos a, \qquad Y_1 = R \sin a.$$

Finally, substituting $x - x'$ for X, etc., in the equations of circle and tangent, we have, in general, co-ordinates, for the circle,

$$(x - x')^2 + (y - y')^2 = R^2,$$

the general equation of a tangent,

$$(x - x') \cos a + (y - y') \sin a = R.$$

This equation contains one arbitrary constant, a, to which any value may be given. Whatever this value may be, the line fulfils the condition of tangency to the given circle. Thus, given the circle $x^2 + y^2 + 2x - 4y - 8 = 0$. Reducing the equation to the form containing x', y' and R, it becomes $(x + 1)^2 + (y - 2)^2 = 13$. Making the substitutions in the general equation of tangent, we have $(x + 1) \cos a + (y - 2) \sin a = \sqrt{13}$, for the equation of *any* tangent to the given circle.

Examples.—Find the general equation of the tangent to the circle $x^2 + y^2 - 6x + 8y = 0$.

Find the equation of a tangent to this circle, parallel to $3y - 4x = 10$, by giving $\cos a$ and $\sin a$ the same values as in this line.

Verify that the line found is a tangent, by finding its intersections with the given circle.

121. To find the equation of a tangent line, in terms of the coordinates of the point of contact, we may eliminate a, introducing X_1 and Y_1, in the equation of tangent.

From the values of X_1 and Y_1, we find

$$\cos a = \frac{X_1}{R}, \qquad \sin a = \frac{Y_1}{R},$$

substituting which the equation of tangent becomes

$$XX_1 + YY_1 = R^2.$$

This is the equation of a line touching the circle at the point P_1, which is supposed to be a point of the curve. Thus, (3, 1) is a point of the circle $x^2 + y^2 = 10$, for it satisfies its equation. Hence $3x + y = 10$ is the tangent to the circle at that point. It is easy to verify, both that this line passes through the given point and that it touches the circle.

Examples.—Find tangents to $x^2 + y^2 = 25$, at the points of the curve whose abscissa is 4.

Ans. The corresponding ordinate is ± 3, and the tangents are $4x + 3y = 25$ and $4x - 3y = 25$.

Find lines touching $x^2 + y^2 = 9$, at the points where it cuts the axis of X.

It must be remembered, however, that the *form* of the above equation does not make P_1 a point of the line. X_1 and Y_1 merely occur in the equation as *constants*, and if they are substituted for the *variables* X and Y, in order to see whether P_1 satisfies the equation, the result is

$$X_1^2 + Y_1^2 = R^2.$$

This is an equation of condition, true *only* when P_1 is a point of the circle. Therefore, $XX_1 - YY_1 = R^2$ is *not* the equation of a tangent *unless* P_1 is on the curve. We shall hereafter see what it represents in case P_1 is not a point of the curve.

122. Since a straight line may be made to fulfil two conditions, a tangent line, which already fulfils one condition, may be made to

fulfil another; for instance, that of passing through a given point. Suppose the equation of the circle to be given in its central form, then we may assume for the tangent the form $X \cos \alpha + Y \sin \alpha = R$, and determine α by an equation of condition. We need not determine the angular value of α, of course, but only determine the coefficients, $\cos \alpha$ and $\sin \alpha$, by the equation of condition, and the relation $\sin^2 \alpha + \cos^2 \alpha = 1$.

We may instead of this assume the form $XX_1 + YY_1 = R^2$, and determine the constants X_1 and Y_1 by the equation of condition expressing that the given point is on the tangent, and the relation $X_1^2 + Y_1^2 = R^2$, which we have seen is the condition necessary to make the line a tangent. Thus, given the circle $x^2 + y^2 = 25$, the equation of a tangent becomes $xx_1 + yy_1 = 25$. Let it be required to pass through (7, 1). This gives the equation of condition $7x_1 + y_1 = 25$. Combining this with $x_1^2 + y_1^2 — 25$, expressing that the point sought is on the circle, we have

$$x_1^2 + 625 - 350x_1 + 49x_1^2 = 25;$$

hence $\qquad x_1^2 - 7x_1 = -12 \qquad$ and $\qquad x_1 = 4$ or 3.

The corresponding values of y_1 from $7x_1 + y_1 = 25$ are -3 and 4, hence there are two solutions, placing the point P_1 at $(4, -3)$ and at $(3, 4)$. By substitution in $xx_1 + yy_1 = 25$, we form the equations of two tangents to the given circle

$$4x - 3y = 25 \text{ and } 3x + 4y = 25,$$

both of which pass through the given point (7, 1).

There are two solutions, because the given point being *without* the circle two tangents may be drawn from it to the circle. If the given point had been *within* the circle, the impossibility of drawing tangents to the circle through it, would have been shown by finding imaginary values for x_1 and y_1. If the point were on the curve, there would be but one solution, equal values being found for x_1, and P_1 coinciding with the given point.

EXAMPLES.—Find the tangents to $x^2 + y^2 = 10$, which pass through $(4, 2)$; tangents to $x^2 + y^2 = 16$, passing through $(4, -4)$; etc., etc.

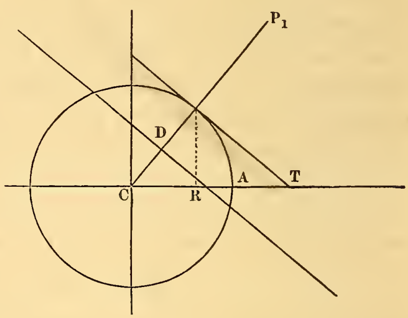
Show that $(2, 2)$ is within the circle $x^2 + y^2 = 10$, by the impossibility of tangents; and that $(-1, 3)$ is on the curve, by finding the *single* tangent that may be drawn through it.

123. Strictly speaking, $XX_1 + YY_1 = R^2$ is not the equation of a tangent, but a *form* of the equation of the straight line, in which a certain relation between the arbitrary constants X_1 and Y_1 is the *condition* of tangency.

Considering X_1 and Y_1 as two *arbitrary* constants, independent of the condition $X_1^2 + Y_1^2 = R^2$, the equation may represent any* line. We may construct it by means of its intercepts upon the axes (which are, in this case, perpendicular diameters).

$$X_0 = \frac{R^2}{X_1}, \qquad Y_0 = \frac{R^2}{Y_1}.$$

If P_1 is a point of the circle, these values show that the intercept of a tangent upon a diameter is a third proportional to the corresponding co-ordinate of the point of contact and the radius. That is, in the figure CR : CA : : CA : CT. Now suppose P_1 to be any point *not* on the curve, and join it with the centre. The equation of the diameter CP_1 is $Y = \frac{Y_1}{X_1} X$, Art. 65; it is therefore perpendicular to $XX_1 + YY_1 = R^2$, in which the direction ratio is $-\frac{X_1}{Y_1}$. Whatever the position of P_1 therefore, the line $XX_1 + YY_1 = R^2$ is perpendicular to the diameter on which P_1 is situated. Its distance from the centre, as found by Art. 53, is



$$\frac{R^2}{\sqrt{X_1^2 + Y_1^2}}.$$

* Or rather, any line not passing through the centre, which is here the origin; since the equation contains an absolute term.

But $\sqrt{X_1^2 + Y_1^2}$ is the length CP_1. Hence the distance of the line from the centre is a third proportional to the distance of P_1, and the radius. Finding the point D, so that CD has this value, we may construct the line perpendicular to CP_1.

If CP_1 is greater than the radius, CD is less; and if CP_1 is less, CD is greater; so that $CP_1 \times CD = R^2$. Hence if P_1 is without the circle, the equation represents a secant line as in the figure, and if P_1 is within the circle, it represents a line not cutting the circle. The line

$$XX_1 + YY_1 = R^2$$

is called the *polar* of the point P_1 *with reference to the circle* $X^2 + Y^2 = R^2$, and a tangent is the polar of its point of contact.

124. The above equation is therefore a *formula* for the equation of a tangent in terms of its co-ordinates of contact; but more generally considered, it is a formula for the *polar* of any point, of which the tangent at a given point of the curve is a special case. The *form* of this equation (which is the central equation of a polar) shows two things independently of all we have hitherto proved. First, that "if one point is on the polar of another point, the latter is also on the polar of the first point." For suppose P_2 to be on the polar of P_1; we have the equation of condition

$$X_2X_1 + Y_2Y_1 = R^2.$$

But this is also the condition that P_1 is on the line

$$XX_2 + YY_2 = R^2,$$

which is the polar of P_2. Points so related are therefore said to be *reciprocally polar.*

Secondly, that a point *polar to itself*, or on its own polar, is also on the circle of reference. For suppose P_1 to be such a point, we have the equation of condition

$$X_1^2 + Y_1^2 = R^2,$$

which also expresses that P_1 is on the circle. This is the condition of tangency pointed out in Art. 121.

We may now state the problem solved in Art. 122 (namely, to find the points of contact of tangents to $x^2 + y^2 = 25$, passing through (7, 1)) in the following manner: To find points which shall be self-polar, with reference to $x^2+y^2=25$, and at the same time polar to (7, 1). The solution amounts to finding points on the circle $x^2+y^2=25$, and also on the polar of (7, 1) which is $7x+y=25$. The intersections of these are $(4, -3)$ and $(3, 4)$, the points of tangency required (whose co-ordinates were represented by x_1 and y_1 in Art. 122).

125. To find the general equation of a polar with reference to any circle, which of course includes the general equation of a tangent, we must substitute in

$$XX_1+YY_1=R^2$$

the general values of the central co-ordinates, both of the variable point P and the fixed point P_1. Thus

$$(x-x')(x_1-x')+(y-y')(y_1-y')=R^2$$

is the general equation, in which x', y' and R^2 are constants derived from the circle, and x_1, y_1 are the co-ordinates of a given point. To adapt this formula to the general equation of the circle, we expand it to

$$xx_1+yy_1-x'(x+x_1)-y'(y+y_1)+x'^2+y'^2-R^2=0,$$

and compare it with the expanded form of the equation of the circle in Art. 107. The coefficients of $(x+x_1)$ and $(y+y_1)$ are one-half those of x and y in the equation of the circle, and the last three terms are equivalent to the absolute term f. Or, $-x'=\frac{1}{2}d$, $-y'=\frac{1}{2}e$ and $x'^2+y'^2-R^2=f$. Hence

$$xx_1+yy_1+\tfrac{1}{2}d(x+x_1)+\tfrac{1}{2}e(y+y_1)+f=0$$

is the general equation of a polar or tangent to the circle,

$$x^2+y^2+dx+ey+f=0.$$

126. It appears from these equations, that having the equation of a circle, the general formula for a polar with reference to that circle may be produced by substituting xx_1 for x^2, yy_1 for y^2, $\frac{1}{2}(x+x_1)$ for x and $\frac{1}{2}(y+y_1)$ for y. Thus, given the circle $x^2+y^2-4x+6y-7=0$, the formula for a tangent or polar is $xx_1+yy_1-2(x+x_1)+3(y+y_1)-7=0$.

Now the point $(6, -1)$ is a point of the circle, because it satisfies its equation; hence the tangent at that point is found by making $x_1 = 6$ and $y_1 = -1$, which gives

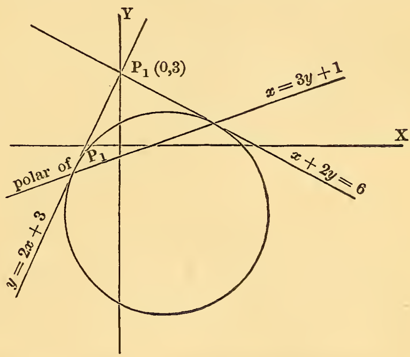
$$6x - y - 2x - 12 + 3y - 3 - 7 = 0,$$
$$\text{or} \quad 4x + 2y = 22, \quad \text{or} \quad 2x + y = 11.$$

That this line is actually tangent to the circle, may be shown by finding its intersection with the circle, which is in the single point $(6, -1)$.

Again, $(0, 3)$ is a point without the circle, from which suppose it required to draw tangents to the circle. Making $x_1 = 0$ and $y_1 = 3$, we shall have the equation of the polar line,

$$3y - 2x + 3y + 9 - 7 = 0,$$
$$\text{or} \quad 6y - 2x + 2 = 0, \quad \text{or} \quad x = 3y + 1.$$

The intersection of this line with the circle is in the two points $(4, 1)$ and $(-2 -1)$. These points are the points of contact for the required tangents. Finally, the tangents at these points, by the same formula, are



$x + 2y = 6$, and $y - 2x = 3$,

both of which pass through the given point $(0, 3)$. In the annexed figure, the circle and lines of this problem are constructed for the sake of illustration.

Examples.—Find the equations of tangents to the circle $x^2 + y^2 + 2x - 2y = 8$, at the four points where it cuts the axes.

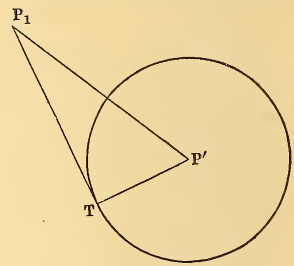
Find tangents to the same circle, passing through $(4, -4)$; through $(-1, 11)$.

Determine a point on the polar of $(1, 1)$; form the equation of its polar and show that it passes through $(1, 1)$.

Show generally, by means of the polar formula, the property of points "reciprocally polar;" also that of "self-polar" points.

Length of Tangent from Given Point.

127. To find the length of the tangents from a given point to a circle, the most obvious method is to find the co-ordinates of one of the points of contact, and to use the formula of Art. 12. But there is a simpler method. For it has been shown that a tangent is perpendicular to the radius at its point of contact; hence, if we join both the given point, and the point of contact, with the centre, we shall form a right-angled triangle, from which it appears that the square of the tangent is the square of the distance of the point from the centre, diminished by the square of the radius. But the square of the distance of P_1 from the centre is



$$P'P_1^2 = (x_1 - x')^2 + (y_1 - y')^2,$$

hence the square of the tangent is

$$P_1T^2 = (x_1 - x')^2 + (y_1 - y')^2 - R^2.$$

Now this expression is the result of substituting the co-ordinates of P_1 for x and y, in the first member of the equation of the circle,

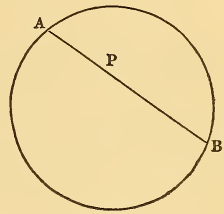
$$(x - x')^2 + (y - y')^2 - R^2 = 0.$$

The value of the expression will be the same, if substitution is made in an equation of the form, $x^2 + y^2 + dx + ey + f = 0$, so that it is not necessary to find the values of x', y' and R^2, but only to make the second member zero, and the coefficients of x^2 and y^2, unity. Thus, in the example of the last Art., where P_1 is the point $(0, 3)$, and the equation of the circle is $x^2 + y^2 - 4x + 6y - 7 = 0$, substitution of $x = 0$, $y = 3$ in the first member gives 20. This of course shows that the point is *not on the circle;* but we now see that it also shows that the point is at such a distance that the tangent is $\sqrt{20}$.

128. We see then, that the *expression*,

$$x^2 + y^2 + dx + ey + f$$

gives the value of the square of the tangent to a certain circle from any point *without* it; and the *equation* formed by putting this expression equal to zero, expresses that the tangent from the point P to this circle is zero, or that the point P is *on* the circle. When the point P is *within* the circle, as in the annexed figure, its distance from the centre is less than the radius, therefore, for such a point, the above expression (which is the square of the distance, minus the square of the radius), is negative. For every point of the plane, this expression has a certain value, which depends upon the position of the point relatively to the circle, and is *independent of the position of the origin.* For the origin itself, the value of the expression is the absolute term; hence for any point, it is the *absolute term which the equation would have if transformed* to that point as origin.* This conclusion is in accordance with the last paragraph of Chap. III.



129. The value of the above expression, for any point, may therefore be called the absolute term for that point. Now in Arts. 112 and 114, it is shown that the absolute term is the product of a secant and its external segment, or of the segments of a chord passing through the origin, according as it is positive or negative. We have now the means of readily computing the value of the absolute term for any point. Thus given the circle

$$x^2 + y^2 + 4x - 2y - 10 = 0,$$

the absolute term for (2, 6) is found by substitution in the first member to be 26; hence 26 is the constant value of the product of a secant drawn through this point, and its external segment. The absolute term for (— 3, 1) is — 14; the negative sign shows that the point is within the circle, and 14 is the constant product of the segments of a chord drawn through this point, as PA . PB, in the figure.

In general, let $x^2 + y^2 + dx + ey + f = 0$ be the equation of the circle, and P, *any* point whose co-ordinates are x and y; and

* Compare also the value of f in Art. 107, $f = x'^2 + y'^2 - R^2$, which shows that the absolute term is the square of the distance of the origin from the centre, minus the square of the radius.

through P let a straight line be drawn cutting the circle in A and B; then

$$x^2 + y^2 + dx + ey + f = \text{PA} \times \text{PB},$$

in which if PA and PB are measured in the same direction from P, they must be regarded as having the same sign and their product as positive; but if they are measured in opposite directions they have different signs, and their product is negative. This equation may be regarded as including the equation of the circle itself, for if we put the product PA . PB equal to zero, one of the segments must be zero, and the point is on the circle. If we assign any other constant value to PA . PB, we have the equation of a concentric circle, because it has the effect of changing the absolute term only, Art. 109. In other words, for all points of a concentric circle this product has the same value. There is no limit to the positive values which may be assigned to it, but the greatest possible negative value is $-\text{R}^2$, which reduces the equation of the concentric circle to an infinitesimal.

EXAMPLES.—Given the circle $x^2 + y^2 - 2x - 4y - 20 = 0$, what is the length of the tangent from (8, 3)? from (6 — 2)? from (0, 7)? etc., etc.

Find the value of PA . PB for the point (2, 2), and whether it is without or within the circle; for (0, 7); for (5, 5); etc.

INTERSECTIONS OF CIRCLES.

130. To find the intersections of two circles, it is, of course, necessary to find values of x and y, which satisfy both their equations. Although the rules of common algebra do not furnish a general method of solving two simultaneous equations of the second degree between x and y, yet if the equations are those of circles, they may be solved in the following manner:

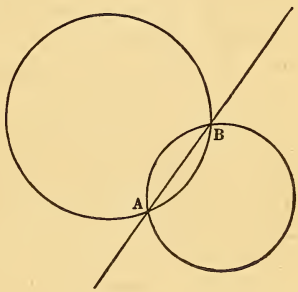
Let $x^2 + y^2 - 2x + y + 1 = 0$, and $x^2 + y^2 - 4x + 3y + 3 = 0$ be the given equations. Subtracting the second from the first, we have an equation of first degree, $2x - 2y - 2 = 0$, which may be combined with either of the given equations giving two solutions; namely, $x = 1$ or $\frac{1}{2}$, $y = 0$ or $-\frac{1}{2}$. Hence the circle represented by these equations intersect in $(1, 0)$ and $(\frac{1}{2}, -\frac{1}{2})$.

As the values of x come from the solution of a quadratic equation, they may be equal, in which case the circles will meet in a single point or *touch* each other; or they may be imaginary, in which case the circles do not intersect.

EXAMPLES.—Find the intersections of $x^2 + y^2 - 3x + 4y = 10$, and $x^2 + y^2 + 6x + y = 7$; of $x^2 + y^2 + 2x + 2y - 27 = 0$, and $x^2 + y^2 - 14x - 10y - 99 = 0$; etc., etc.

131. We solved the above problem by the aid of an equation of first degree, which was formed by combining the given equations. In other words, we employed the equation of a straight line, which, according to Art. 41, passes through the points of intersection sought, and then we found the intersections of this line with one of the circles.

In general, let the circles in the figure have for their equations,



$$x^2 + y^2 + dx + ey + f = 0$$

and $$x^2 + y^2 + d'x + e'y + f' = 0,$$

then the equation of the straight line passing through the intersections, A and B, is

$$(d - d')x + (e - e')y + (f - f') = 0.$$

If the circles intersect, as in the figure, AB is a *common chord.* If the circles touch, A and B coincide, and the line is a common tangent. If the circles do not meet, the line does not cut either of them; but it still has a definite position, and if its equation be combined with each of given equations, there will result the same *imaginary values* for the co-ordinates of intersection; and therefore the line is said to meet the two circles in the same *imaginary points.*

This line, whether cutting the circles or not, is called the *radical axis* of the two circles.

132. The intersections of two circles are thus determined by the intersection of a certain straight line with one of them. Therefore, in accordance with the language adopted in Art. 117, two circles are said to meet in two *real*, *coincident* or *imaginary* points. But if the circles are concentric—that is, if the equations differ only in their absolute terms (Art. 109), $d = d'$ and $e = e'$, and the equa-

tion of the radical axis reduces to the impossible form, *a constant* $= 0$. Thus, given the equations $x^2 + y^2 - 2x + y + 1 = 0$, and $x^2 + y^2 - 2x + y - 5 = 0$; subtracting one from the other, we have the impossible result $6 = 0$. This, of course, indicates that the circles do *not* intersect, just as a similar result did of the lines $x + y = 1$ and $x + y = 3$, in Art. 40. But as parallel lines were said, in Art. 42, to meet at *infinity*, and the impossible result (of the form $C = 0$) was considered as the equation of a line at infinity, so we shall say that the radical axis of concentric circles is the line at infinity, and therefore that these circles meet at infinity.

133. In the case of parallel lines, which, though they do not intersect, are said to intersect in points infinitely distant, we may interpret the expression as follows: When two straight lines are *very near and approaching* to parallelism, their point of intersection is at a *very great and increasing* distance. And this increase goes on *without limit*, until when the lines are actually parallel, the point ceases to exist, and the distance is said to be *infinitely* great. The present case may be explained thus: When the circles are nearly concentric, the values of the coefficients $(d - d')$ and $(e - e')$ in the equation of the radical axis are very small, and therefore the radical axis is very distant. (This is easily seen by considering its intercepts.) Finally, when the circles are actually concentric, the radical axis is infinitely distant. Now when this line was at a great distance from the circles its intersections with them were of course imaginary. But when the line is at infinity they must be considered *infinite* as well as *imaginary*.

In speaking of the actual intersections of lines we should have to say: Straight lines *may* meet in one point but not more; circles *may* meet in two points but not more. But admitting the expressions, infinite and imaginary points, we may say: Straight lines always meet in one point, circles always meet in two points.*

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* The real ground on which we use the expressions point at infinity, imaginary points, etc., in cases where intersections do not exist, is that the general values of the co-ordinates of intersection take these special forms in certain cases. If general values were found for two circles in the manner we have used in the numerical examples, $d = d'$ and $e = e'$ would make them infinite. If two equations of second degree were combined in *general* form, there would be *four* solutions, two of which become *infinite and imagi-*

Combined Equations of Circles.

134. If the equations of two circles be combined according to the principle explained in Art. 41, we shall have the equation,

$$x^2 + y^2 + dx + ey + f + k(x^2 + y^2 + d'x + e'y + f') = 0,$$

representing a line passing through both their intersections, whatever be the value of k.

Now since the coefficients of x^2 and y^2 are the same—namely, $1 + k$, this equation represents a circle; except when $k = -1$, and x^2 and y^2 disappear from the equation. In that case, the equations are simply subtracted, one from the other, and the result is the equation of the radical axis. The combined equation, in which k is an arbitrary constant, represents therefore a series of circles passing through the intersections of the given circles. Thus the combined equation of the circles of Art. 130 is $x^2 + y^2 - 2x + y + 1 + k(x^2 + y^2 - 4x + 3y + 3) = 0$. This represents a series of circles passing through $(1, 0)$ and $(\frac{1}{2}, -\frac{1}{2})$, which we found to be the intersections of these circles; and the equation $2x - 2y - 2 = 0$, which we used in finding the intersection, was but a particular case of the combined equation. Let $k = 1$, and we have (dividing by 2) $x^2 + y^2 - 3x + 2y + 2 = 0$. That this circle intersects both of the given ones in the points $(1, 0)$ and $(\frac{1}{2}, -\frac{1}{2})$ is verified by the fact that, in connection with either of the given circles, it has the radical axis $x - y - 1 = 0$, which is the same as $2x - 2y - 2 = 0$, the radical axis of the original equations.

135. As k is an arbitrary constant, it may be determined by an equation of condition, so as to make the circle pass through a third point. Thus, let the circle be required to pass through $(1, 2)$, as well as the points $(1, 0)$ and $(\frac{1}{2}, -\frac{1}{2})$, through which it must pass, whatever the value of k. Substituting $x = 1$ and $y = 2$ in

$$x^2 + y^2 - 2x + y + 1 + k(x^2 + y^2 - 4x + 3y + 3) = 0.$$

we have the equation of condition, $6 + 10k = 0$; hence $k = -\frac{3}{5}$. With this value of k, the equation reduces to

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nary for circles, while the other two *may* be real. But for concentric circles all four become infinite and imaginary.

$$x^2 + y^2 + x - 2y - 2 = 0.$$

We have here another proof of what was shown in Article 110—namely, that in general, a circle can be found passing through three given points. But if the three points had been in the same straight line, we should have found $k = -1$, and the equation would have reduced to that of the radical axis.

EXAMPLES.—Give the combined equation of the circles, $x^2 + y^2 - 3x + 4y = 10$ and $x^2 + y^2 + 6x + y = 7$.

Determine k so as to make the circle pass through (3, 1); (1, 4); etc., etc.

136. When the equations combined are those of circles which really intersect, as in the last Article, the whole series of circles produced by giving different values to k, have two common points, which determine a common chord or radical axis. That is to say *any two* of the system of circles,

$$x^2 + y^2 + dx + ey + f + k(x^2 + y^2 + d'x + e'y + f') = 0,$$

will give for their radical axis the same line,

$$(d - d')x + (e - e')y + (f - f') = 0.$$

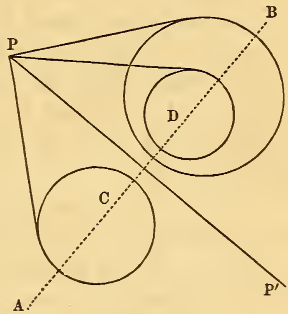
Again, if the original circles touch, the combined equations represent a system of circles, any two of which will determine the same radical axis, or common tangent. But even when the original circles do not meet, the system of circles represented by the combined equation has a *common radical axis*, which is a *real* straight line, and the only one, whose equation is satisfied by the common *imaginary* values of x and y, which satisfy the equations of all the circles. Thus if the system of equations is satisfied by the common imaginary values, $x = 5 \pm 2\sqrt{-1}$ and $y = 3 \pm 4\sqrt{-1}$ the common radical axis is $y = 2x - 7$, for this is readily seen to be the only equation of the form $y = mx + b$, which these values of x and y will satisfy.

Thus, whether the two points common to the system are real, coincident or imaginary, they determine a common radical axis.

137. It was shown in Article 127, that the first member of the equation of a circle is the expression for the square of the tangent from any point to the circle. Hence the combined equation,

$$x^2+y^2+dx+ey+f+k(x^2+y^2+d'x+e'y+f')=0,$$

expresses that the square of the tangent from P on one of the given circles, equals the square of the tangent upon the other, multiplied by $-k$. Suppose $k=-1$ (so that $-k=1$), then the tangents from P on the two circles will be equal, as in the figure. But when $k=-1$, the equation becomes that of the radical axis; therefore P is a point of the radical axis, and that line is the locus of the point from which the tangents on the given circles are equal. From this property, it is easy to see, that the radical axis must be perpendicular to the line joining the centres of the circles, as PP′ in the figure.



Now we saw above, that the whole system of circles, produced by giving k different values, has a common radical axis; therefore the tangents from P to *any pair* of circles, and consequently to *all* the circles in the system, are equal. Hence also, the centres of all the circles must be on the same straight line, AB; for the line joining the centres of any pair of the circles is perpendicular to the radical axis.

When, as in the figure, the circles do not intersect, the distance of any point, P, from AB is less than the common length of the tangents from P, and there may be found two points C and D on AB, at a distance from P, equal to the tangents. These points are infinitesimal circles belonging to the system. (See Art. 108.) If their equations be combined, and a proper value be given to k, we may form the equation of any circle of the system; while, if we make $k=-1$, we shall still have the equation of PP′. Hence PP′ is the locus of a point equidistant from C and D, and each of the circles is the locus of a point whose distances from C and D are in a constant ratio.

Examples.—Find a point from which the tangents to the three circles, $x^2+y^2-2x-6y+9-0$, $x^2+y^2-4x+4y-1=0$ and $x^2+y^2+6x-2y+6=0$, shall be equal. (The point is on the radical axis of the first and second circles, and also on that of the first and third.)

Show generally, that the three radical axes, of three circles taken in pairs, meet in a point (forming the general equation of the line passing through the intersection of two of the radical axes by Art. 41, we shall find that the third radical axis is such a line).

Prove from its equation, that the radical axis of any two circles is perpendicular to the line joining their centres.

What is the locus of a point from which the tangent on one of two given circles is double that on the other?

What is the result of giving a positive value to k in the combined equation of two circles?

Ans. A circle, every point of which is without one and within the other of the given circles.

State a geometrical property of all the points of the circle corresponding to $k=1$, by the help of the interpretation of the first member given in Art. 129 and show that its centre is midway between those of the given circles. (This circle is only possible when the given circles intersect, as in Art. 131.)

Discuss the equation $x^2+y^2+dx+ey+f+k\,(\mathrm{A}x+\mathrm{B}y+\mathrm{C})$.

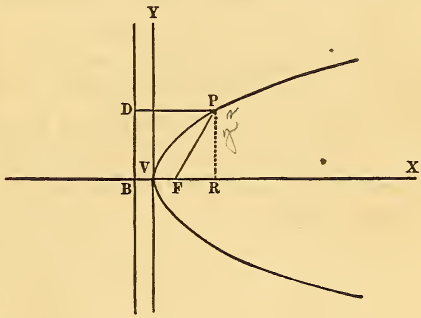
Give a general equation for the circle passing through two given points.

CHAPTER V.

THE PARABOLA.

138. If a point move in such a manner that its distances from a fixed point and a fixed line shall be constantly equal, it will describe a curve which is called a *parabola.* The fixed point is called the *focus*, and the fixed line the *directrix.*

To find the equation of the parabola, that is, of the locus of the point P moving according to the above law, assume as the axis of X a line passing through the focus and perpendicular to the directrix; and let the origin be the point V, midway between the focus and directrix, and the axis of Y, parallel to the directrix. Denote the distance, FB, from the focus to the directrix, by p, and draw the ordinate PR, and the lines PF and PD, which by the definition of the parabola must be equal. By the right-angled triangle PRF, $PR^2 = PF^2 - FR^2$; but $PF = PD = BR$, and hence is represented by $x + \frac{1}{2}p$, and FR is $x - \frac{1}{2}p$; since BV and VF are each one-half of BF or p. Therefore $y^2 = (x + \frac{1}{2}p)^2 - (x - \frac{1}{2}p)^2$, or expanding and reducing,



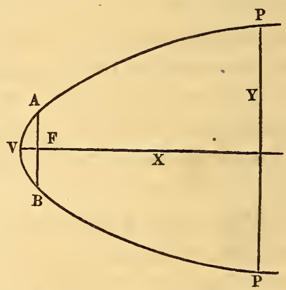
$$y^2 = 2px.$$

The form of this equation shows that the curve passes through the origin, and that it is symmetrical with respect to the axis of X, as we should expect from the manner in which the origin and axes were chosen. It is the simplest form of the equation of a parabola; for this reason, VX is called the *axis of the curve*, and V the *vertex*. To indicate that the curve is referred to its axis and vertex, we may use X and Y in this equation. When the properties of the parabola are investigated *geometrically*, the lines VR and PR are defined as the abscissa and ordinate of P. We may therefore call these lines the *geometrical abscissa and ordinate* of P, and

$$Y^2 = 2pX,$$

the *geometrical equation* of the parabola.

139. A chord perpendicular to the axis of the parabola, as PP in the figure, is called a *double ordinate*. It is bisected by the axis, and its halves are the equal positive and negative ordinates corresponding to the same abscissa. The double ordinate passing through the focus is called the *parameter*. To find its value, substitute for X the abscissa of the focus, which is $\frac{1}{2}p$, we obtain $Y^2 = p^2$ or $Y = \pm p$, hence the parameter is $2p$, or four times the distance from the vertex to the focus. In the equation, $2p$, the value of the parameter, is the coefficient of the geometrical abscissa; and the equation expresses that the square of an ordinate equals the product of the parameter into the corresponding abscissa; or that the ordinate of a point is a mean proportional between the parameter and its abscissa, AB or $2p : Y :: Y : X$.



If the abscissa increase uniformly, the ordinate likewise increases, but at a rate which becomes slower and slower; nevertheless, the ordinate may be increased to any extent, if we allow the abscissa to increase without limit. The curve is unlimited in extent, like the lines represented by equations of the first degree, and unlike the circle, which is a *closed* curve or a line returning into itself. Two parabolas may be compared by means of their parameters, just as two circles are by means of their radii; thus, $y^2 = 5x$ and $y^2 = 2x$ having the parameters 5 and 2, differ just as $x^2 + y^2 = 25$ and $x^2 + y^2 = 4$, which have respectively the same numbers of units for radii. Hence parabolas are considered as differing in *size* and not in *shape.* The figures only represent small portions of parabolas taken near their vertices, and the *apparent* shape of the curve depends upon the extent of the portion drawn. Thus in the last two figures different extents of the same parabola are drawn, the parameters being the same.

140. If the vertex of a parabola is situated, not at the origin, but at P′, the axis of the curve being parallel to the axis of X, the geometrical abscissa and ordinate, X and Y, will be denoted by $x - x'$ and $y - y'$, and the equation becomes

$$(y - y')^2 = 2p\,(x - x').$$

This equation contains two new constants, just as the equation of a circle in any position contains two more constants than the central equation. However, this is not a formula for *any* parabola, but for a parabola having a given vertex and parameter, and its axis *parallel to the axis of* X. Thus, if the vertex is to be at $(2, -1)$ and the parameter 4, the equation of the parabola is $(y + 1)^2 = 4\,(x - 2)$ or $y^2 + 2y = 4x - 9$.

Examples.—Give the equation of the parabola with a parameter 8, and vertex at (2, 6); parameter 3, and vertex $(-2, 2)$, etc., etc.

141. If in place of $2p$ we put a negative quantity, the equation represents the same parabola, but extending from the vertex toward the left. Thus, $y^2 = 8x$ and $y^2 = -8x$ are the equations of equal parabolas, the first extending toward the right and the second toward the left from the origin. For whatever values of y are given by a positive value of x in the first, the same values of y are given by a like negative value of x in the second; thus $(2, \pm 4)$ $(8, \pm 8)$ are points on the first, (-2 ± 4) (-8 ± 8) are points on the second. The first contains no points on the left of the origin, the second no points on the right.

With a given vertex and a given point on the curve, $2p$ may be determined by an equation of condition, and its sign will show in which direction the curve extends. In fact, since $2p$ is four times VF, its sign is the same as that of VF, and therefore shows whether the focus (and hence the curve) is on the right or left of the vertex.

EXAMPLES.—Give the equations of parabolas extending to the left, equal to $y^2 = 6x$, and with the vertices (2, 1), (—1, 3), (0, 6), etc., etc.

Determine the parabola with vertex (5, 2) and passing through (1, 1), etc.

POLAR EQUATIONS.

142. Using the formulæ of transformation from rectangular to polar co-ordinates, in the geometrical equation, we have

$$r^2 \sin^2 \theta = 2pr \cos \theta,$$

which is satisfied by $r = 0$, whatever the value of θ. Dividing by r and reducing, we find the other value of r to be

$$r = \frac{2p \cos \theta}{\sin^2 \theta}.$$

This is the equation of the parabola when the vertex is the pole and the axis of the curve the initial line. If $\theta = 0°$, the value of r is infinite. If $\theta = 90°$, $r = 0$; between these values r is positive. When θ is in the second or third quadrant, r is negative because $\cos \theta$ is negative, and $\sin^2 \theta$ is always positive. From $\theta = 90°$ to $\theta = 180°$, the lower half of the curve is described by negative values of r, and for each succeeding 180° the curve is repeated. The line $\theta = 90°$, which was the axis of Y, is tangent to the curve, because it makes $r = 0$; that is, does not meet the curve except at the pole.

143. The polar equation when the focus is the pole takes a more simple form. It may be found directly from the figure of Art. 138: thus, $PF = r$ and $FR = x$, as referred to the focus; but from the definition, $PF = BR$, or $r = x + p$. This relation between r and x, which expresses the definition directly, is, in fact, an equation of the parabola, but since $x = r \cos \theta$, the true polar equation is $r = r \cos \theta + p$, or

$$r = \frac{p}{1 - \cos \theta}.$$

This equation gives positive values of r, for all values of θ, because $\cos \theta$ is always less than 1. $\theta = 0$ gives $r = \infty$. $\theta = 90°$

gives $r = p$, half the parameter or perpendicular chord through the focus. $\theta = 180°$ gives $r = \frac{1}{2}p$, the distance from focus to vertex.

EXAMPLES.—Give the equation of the parabola whose parameter is 10, the focus being the pole.

Give the values of r corresponding to $\theta = 60°$, $\theta = 90°$, $\theta = 120°$.

144. The general polar equation of the parabola, as of the circle, is of the second degree, and each value of θ gives two values of r. The first of the two special equations we have just found, is of the first degree for the reason given in Art. 115, that we neglect one of the values of r which corresponds to every value of θ, because the pole was taken on the curve. In the last equation we also neglect one of the values of r corresponding to each value of θ; namely, the negative value. The double value of r may be found by following the regular method of transformation. Thus, the equation $y^2 = 2px$ transformed to the focus $(\frac{1}{2}p, 0)$, is $y^2 = 2px + p^2$.* Transforming to polar co-ordinates, we have

$$r^2 \sin^2 \theta = 2pr \cos \theta + p^2.$$

This quadratic for r may be solved by adding $r^2 \cos^2 \theta$ to each member, which gives $r^2 = (r \cos \theta + p)^2$, hence

$$r = \frac{p}{1 - \cos \theta} \text{ or } \frac{-p}{1 + \cos \theta}.$$

The latter value of r is always negative, and for a given value of θ, is numerically the same as the positive value corresponding to $\theta + 180°$. If we give θ all values between 0° and 360°, the whole curve will be described by the positive value of r.

145. A line drawn through the focus and terminated both ways by the curve is a *focal chord.* Its length may be found, in terms

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* This is the equation of a parabola having its *vertex* at the point $(-\frac{1}{2}p, 0)$, on the *left* of the origin (which is to be the focus), and might have been derived from the formula $(y - y')^2 = 2p(x - x')$. We may move the origin to the right by transformation, or the vertex to the left by the formula.

of its inclination to the axis, by adding the *numerical values* of the positive and negative radii vectores corresponding to one value of θ; thus,

$$\frac{p}{1-\cos\theta}+\frac{p}{1+\cos\theta}=\frac{2p}{1-\cos^2\theta}=\frac{2p}{\sin^2\theta}.$$

When $\theta=0°$, this expression becomes infinite. When $\theta=90°$, it becomes $2p$, the parameter, and this is the smallest possible focal chord.

EXAMPLES.—Find the focal chords having inclinations, 60°, 45° and 30°.

Find the chords passing through the vertex, with these inclinations.

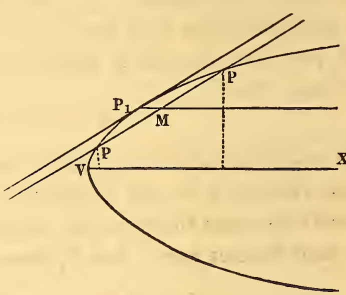
Show that the radius vector from the focus always exceeds the parallel radius vector, or chord, from the vertex.

SECANT AND TANGENT LINES.

146. In finding general expressions for the intersection of a straight line with a parabola, we shall use for the equation of the straight line the form

$$X=nY+a,$$

in which, X and Y are co-ordinates referred to the vertex and axis of the curve, and n is the cotangent of the inclination of the line (Art. 46). Substituting the value of X, in the geometrical equation,



$$Y^2=2pX,$$

we have

$$Y^2-2npY=2pa,$$

a quadratic of which the roots are the ordinates of the points P,P. Completing the square, etc., we obtain

$$Y=np\pm\sqrt{n^2p^2+2pa}.$$

The corresponding values of X, from $X=nY+a$, are

$$X = n^2p + a \pm n\sqrt{n^2p^2 + 2pa}.$$

When the radical $\sqrt{n^2p^2 + 2pa}$ is real, the line is a secant, as PP in the figure; and the rational parts of the above values are the co-ordinates of M, the middle point of the chord PP. The ordinate of M is therefore np; it is independent of a, and is the same for all lines in which the value of n is the same; that is, for all lines having the *same inclination*. Draw the line MP_1 parallel to the axis; it contains all the points whose ordinates are np, hence it is the locus of the middle point M, when the line PP moves with a constant inclination; or it *bisects a system of parallel chords*. Such a line is called a *diameter*, therefore the diameters of a parabola are parallel to the axis.

147. If the radical part of the values of X and Y is zero, the line is a tangent, the points P,P coinciding. Hence the condition of tangency is $n^2p^2 + 2pa = 0$. When the constants, n and a, which determine the position of the line, are so related as to satisfy this equation, the line is tangent to the parabola. Thus, if the parameter is 8, or $p = 4$, the line $x = 2y - 8$, in which $n = 2$ and $a = -8$, is a tangent, as may be verified by finding its intersection with $y^2 = 8x$.

To derive the equation of a tangent having a given inclination, we must determine a in terms of n, from the condition of tangency, and put the value found, which is $a = -\frac{1}{2}n^2p$, in the equation of the line; thus

$$X = nY - \tfrac{1}{2}n^2p.$$

The co-ordinates of the point of contact are found by giving the same value to a, in the co-ordinates of intersection, X and Y, or what is the same thing in the co-ordinates of M, because the radical parts become zero. Let P_1 denote the point of contact, then

$$X_1 = \tfrac{1}{2}n^2p \qquad \text{and} \qquad Y_1 = np.$$

These values satisfy both the equation of the curve and the equation of the tangent.

148. If we suppose the value of n to be the same that it was for the secant line PP, in the figure, the ordinate of P_1 is the same as that of M, hence *the extremity of the diameter is the point of contact for a tangent parallel to the chords it bisects.* P_1 is called the

vertex of the diameter P_1M. The axis is one of the diameters, and its vertex V is called the *principal vertex.* A diameter may be constructed, so as to bisect chords drawn in a given direction, and then a line drawn in the same direction through its vertex, will be a tangent. Thus we can construct geometrically a tangent parallel to a given line.

For the equation of tangent parallel to a given line, give to n the value it has in the given equation, and for a perpendicular tangent, make n the negative of the reciprocal of n in the given equation. Thus, given the parabola $y^2 = 8x$, in which $p = 4$, the formula for a tangent becomes $x = ny - 2n^2$. Let the given line be $3y - 4x = 20$ or $x = \frac{3}{4}y - 5$; then for a parallel $n = \frac{3}{4}$, hence $x = \frac{3}{4}y - 1\frac{1}{8}$; for a perpendicular tangent $n = -\frac{4}{3}$, hence $x = -\frac{4}{3}y - 3\frac{5}{9}$.

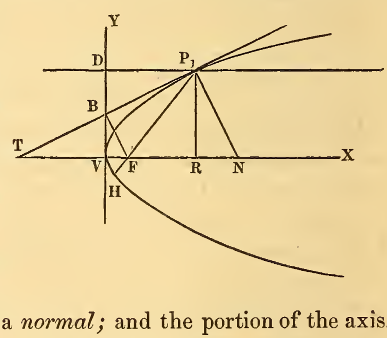
EXAMPLES.—Give the equations of tangents to $y^2 = 6x$, parallel and perpendicular to $2x + 5y + 7 = 0$.

Prove that perpendicular tangents always intersect in a point on the directrix (the equation of the directrix is $x = -\frac{1}{2}p$).

Prove that a perpendicular to a tangent, drawn through the focus, $(\frac{1}{2}p, 0)$, intersects it on the axis of Y.

PROPERTIES OF PARABOLA AND TANGENT.

149. Let T be the point in which the tangent at P_1 cuts the axis. The equation of the tangent gives for the intercept $X_0 = -\frac{1}{2}n^2p$, which is the negative of the value of X_1 the abscissa of P_1; that is, $VT = VR$ and is measured in the opposite direction. Hence TR, which is called the *subtangent, is bisected at the vertex.*



The line P_1N, drawn perpendicular to the tangent at the point of contact, is called a *normal;* and the portion of the axis, RN, is called the *subnormal.* By similar triangles $TR : P_1R :: P_1R : RN$, or, since we have just shown that $TR = 2X_1$, $Y_1^2 = 2X_1 \times RN$. But $Y_1^2 = 2pX_1$ because P_1 is a point of the curve.

Therefore $RN = p$, or the *subnormal is constant and equals half the parameter.*

150. Since $VF = \frac{1}{2}p$, $TF = X_1 + \frac{1}{2}p = \frac{1}{2}TN$, hence a perpendicular FB to the tangent bisects TP_1 by the similar triangles TFB, TNP_1. But the axis of Y also bisects TP_1, and therefore cuts it in the same point, B. It follows that the right triangles FBT, FBP_1 are every way equal, and the angle $FP_1T = FTP_1$; that is, the tangent makes equal angles with the axis and line joining the focus and point of contact, or *bisects the angle between this line and the diameter produced.*

The inclination of the tangent is therefore one-half the angle FP_1D. Let θ represent the inclination of the focal radius vector, P_1F, $\frac{1}{2}\theta$ will then represent the inclination of the tangent at P_1. Then as θ varies from 0° to 360°, this inclination will vary at half the rate, and from 0° to 180°. Consequently, for two values of θ differing by 180°, the inclinations of the tangent will differ by 90°, or the *tangents at the two extremities of a focal chord are perpendicular.*

A tangent at a given point of the curve is constructed (after the axis is drawn) by means of the property of the subtangent. The parameter focus, etc., may then be found by the property of the subnormal. If the curve only was given, we should draw a diameter by bisecting parallel chords; and then a parallel line, bisecting a perpendicular chord, will be the axis.

EXAMPLES.—Prove that $P_1F \times FH = P_1H \times VF$.

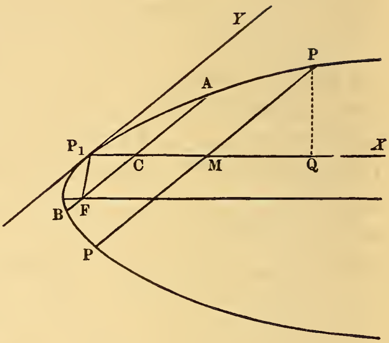
Show that TN equals half the focal chord parallel to P_1T.

OBLIQUE CO-ORDINATES OF PARABOLA.

151. Since every diameter of the parabola bisects chords parallel to the tangent at its vertex, the simplest oblique equations of the parabola will be those which express the relation between these semi-chords and parts of the diameter cut off. Thus, let P_1X and P_1Y, a diameter and tangent at its vertex, be taken as the co-ordinate axes, then the two halves of any chord, PP (which is bisected at M), will be equal positive and negative ordinates, corresponding to P_1M, as an abscissa.

In Art. 146, we found the geometrical co-ordinates of P,P in terms of the constants, n and a, to be

$$X = n^2p + a \pm n\sqrt{n^2p^2 + 2pa},$$
$$Y = np \pm \sqrt{n^2p^2 + 2pa},$$



of which the rational parts are the co-ordinates of M, and the radicals are MQ and PQ, the differences of the co-ordinates of M and P. By the right-angled triangle, PQM, the sum of the squares of these quantities is the square of PM. Again P_1M is the difference of the abscissas of M and P_1, and the latter was found, in Art. 147, to be $X_1 = \frac{1}{2}n^2p$. Therefore, denoting P_1M and PM the oblique co-ordinates by X and Y,

$$Y^2 = (1 + n^2)(n^2p^2 + 2pa) \qquad \text{and} \qquad X = \tfrac{1}{2}n^2p + a.$$

These equations give us values of Y and X in terms of n and a. Now n is constant, because the chord PP is to be always parallel to P_1Y, and n determines the direction of the line PP; if then we can establish a relation between Y and X, which is *independent* of a, it will be true for every position of the point P, and therefore will be the oblique equation of the curve. Now, if we multiply the above value of X by $2p\,(1 + n^2)$, we shall have the value of Y^2, hence the required equation is

$$Y^2 = 2p\,(1 + n^2)\,X.$$

152. The form of this equation is the same as that of the rectangular equation first found, and expresses that the square of the ordinate is a mean proportional between the abscissa and the constant, $2p\,(1 + n^2)$, which may be called an *oblique parameter.* If we denote this quantity by $2p'$, the equation becomes

$$Y^2 = 2p'X.$$

The oblique parameter is always greater than $2p$, which we shall call the *principal parameter* of the parabola. The factor $1 + n^2$

is the square of the cosecant of the inclination of the new axis of Y, because n is its cotangent. If θ denotes this angle,

$$2p' = 2p \operatorname{cosec}^2 \theta = \frac{2p}{\sin^2 \theta};$$

but this is the value of the focal chord making the inclination θ, Art. 145, therefore the parameter for any oblique ordinates is the parallel focal chord, as AB in the figure. The part of the diameter, P_1C, cut off by AB, or abscissa corresponding to the ordinate p', is easily shown from the equation to be $\frac{1}{2}p'$, or one-fourth of the parameter, as in the rectangular equation. The distance P_1F from the vertex to the focus is also one-fourth of the parameter, for it is the same as the distance of P_1 from the directrix, or $X_1 + \frac{1}{2}p = \frac{1}{2}(n^2 + 1)p$.

153. We may now drop the distinction between oblique and rectangular co-ordinates, and regard $Y^2 = 2pX$ as the equation of a parabola having the axis of X as a diameter, the axis of Y a tangent at the origin, and $2p$ for the parameter parallel to the axis of Y. For any parabola whose axis is parallel to the axis of X, let P' be the point of contact of a tangent parallel to the axis of Y, and let $2p$ denote the corresponding parameter, then the equation is

$$(y - y')^2 = 2p(x - x').$$

Thus $(y + 1)^2 = 4(x - 2)$ represents a parabola, whatever the inclination of the co-ordinate axes; $(2, -1)$ is a point of the curve (as easily verified) and 4 is the parameter parallel to the axis of Y. If the inclination of the axes is given, the principal parameter may be derived from this, by the relation between parameters, in the last Article. For instance, if the inclination is 60°, since $\sin^2 60° = \frac{3}{4}$, the principal parameter is 3.

154. Expanding the above equation, we have

$$y^2 - 2px - 2y'y + y'^2 + 2px' = 0,$$

which contains only one term of the second degree, y^2. Hence

$$Cy^2 + Dx + Ey + F = 0$$

is the general equation of the parabola with its axis parallel to the axis of X, and

$$y^2 + dx + ey + f = 0$$

is the form which it takes, when we divide through by C, the coefficient of y^2. From the values of d, e and f, in a given equation, the values of $2p$, x' and y'; that is, the parameter and the position of the vertex, may be determined. For, comparing the equation with the expanded form, we find $d = -2p$, $e = -2y'$, $f = y'^2 + 2px'$. Hence

$$2p = -d,\ y' = -\tfrac{1}{2}e \text{ and } x' = \frac{f - y'^2}{2p} = \frac{f - \frac{1}{4}e^2}{-d} = \frac{e^2 - 4f}{4d}.$$

Thus, the equation $8x + 2y^2 + 10 = 4y$ represents a parabola. Arranging the terms and dividing, $y^2 + 4x - 2y + 5 = 0$, in which $d = 4$, $e = -2$ and $f = 5$; substituting which in the values of $2p$, x' and y', we find that the parameter is -4, and the vertex is the point $(-1, 1)$. The negative value of the parameter indicates that the parabola extends from the vertex to the left, as explained in Art. 141.

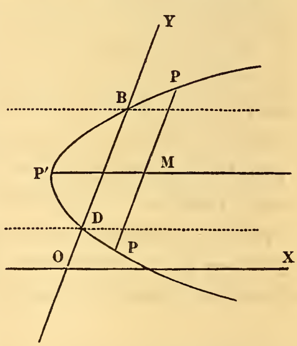
Examples.—Determine the parameter and vertex of $6x - 2y^2 = 4y + 3$; of $5 - y^2 = x + 2y$; of $y^2 + 2x = 0$; etc., etc.

Equations of Parallel Lines.

155. Solving the equation of the parabola as a quadratic for y, we obtain

$$y = -\tfrac{1}{2}e \pm \sqrt{\tfrac{1}{4}e^2 - f - dx},$$

two values of y in terms of x. The rational part is the value of y', and $y = -\frac{1}{2}e$ is the equation of a diameter of the curve, because it expresses the value of the ordinate of M, midway between the points of the curve corresponding to the same abscissa. The radical part is the value of MP, which must be added to and subtracted from the ordinate of M, to produce the ordinates of the points P,P, on the parabola. This radical contains x; it is the variable part of the ordinates, and for certain values of x will be imaginary. The value which makes it zero will be found to be the same as that of x' in the last Article; it is a limit-



ing value of x, because the value of y is only possible when $-dx$ (or $2px$) is algebraically greater than $-(\frac{1}{4}e^2 - f)$.

In the figure, the parabola is so drawn that $-dx$ algebraically increases as x increases; that is, d is negative, and the expression for the parameter is positive. If the parabola extended from P′ toward the left, d would be positive and the parameter negative.

156. If $d = 0$, the term containing x disappears from the equation, and from the radical in the values of y. Hence the equation $y^2 + ey + f = 0$ expresses that y has one of the two constant values

$$y = -\tfrac{1}{2}e \pm \sqrt{\tfrac{1}{4}e^2 - f}.$$

If now $\frac{1}{4}e^2 > f$, these values are real, and the equation is equivalent to two equations of the form $y = b$, therefore it represents *two straight lines parallel to the axis of* X. If $\frac{1}{4}e^2 = f$, the two values of y become identical, and the equation gives but one value of y, $y = -\frac{1}{2}e$, for every value of x; but as the two parallel lines in this case become one, it is called the equation of *two coincident lines.* Finally, if $\frac{1}{4}e^2 < f$, the values of y are imaginary, and the equation is said to represent two *imaginary parallel lines;* for, though satisfied by no real points, the two imaginary values of y are constant. Therefore the equation of the parabola, when we admit zero among the possible values of the constants, may represent two *parallel*, *coincident* or *imaginary lines.*

EXAMPLES.—What lines does $y^2 + 4y = 0$ represent? $2y^2 + 4y + 2 = 0$? $y^2 = 9$? $4y - y^2 - 5 = 0$?

157. The constant values of y, in the last Article, are also the intercepts of $y^2 + dx + ey + f = 0$ on the axis of Y, as seen by making $x = 0$ in the variable values of Art. 155. Let the parabola cut the axis in the two points, B and D; that is to say, suppose the intercepts real, or $\frac{1}{4}e^2 - f$ (the quantity under the radical sign) positive. The position of the points B and D (which depend upon the intercepts) and that of the line P′M (which depends upon y' or $-\frac{1}{2}e$) is not affected by any change in the value of d. But from the values of x' and $2p$, which are

$$2p = -d, \qquad x' = \frac{e^2 - 4f}{4d},$$

it appears that as we diminish d to zero, the parameter decreases without limit, and the distance of the vertex increases without limit. Since $\frac{1}{4}e^2 - f$ is positive, the numerator of x' is positive, and its sign is the same as that of d, or opposite to that of the parameter. In the figure $2p$ is positive and x' negative, so that all values of x, *algebraically greater than* x', give real values of y. Finally, when $d = 0$, the parameter becomes zero, the vertex P′ disappears, because x' becomes infinite; and *all* values of x give real values of y, equal to the intercepts. The curve is then said to *vanish into* the parallel lines passing through B and D.

158. If $\frac{1}{4}e^2 - f$ were negative instead of positive, the curve would not cut the axis of Y, because the intercepts would be imaginary. In that case x' would have the same sign as $2p$, or would be positive for a parabola extending toward the right, like that of the figure. So that when $d = 0$ and x' becomes infinite, there are *no* values of x algebraically greater than x', and hence no values which give real values of y. The curve, therefore, *vanishes into imaginary lines.*

If $\frac{1}{4}e^2 - f = 0$, the intercepts are real and equal, and $x' = 0$; that is, the curve touches the axis of Y, and the vertex P′ is a fixed point on that axis. In this case, real points occur only on the right of the axis when d is negative, and only on the left, when d is positive. But when $d = 0$, x' takes the form $\frac{0}{0}$, the position of the vertex becomes indeterminate, and every value of x gives equal values of y. The curve, therefore, vanishes into a pair of lines coincident with $y = -\frac{1}{2}e$.

In general, as the parameter of a parabola changes sign, passing through the value zero, x' changes sign, passing through the value infinity, and the vertex reappears on the other side of the axis of Y. But in this last case, the vertex is *on* the axis, except in the vanishing case, when it is *anywhere* on the line $y = -\frac{1}{2}e$.

159. An equation of the general form

$$Cy^2 + Dx + Ey + F = 0,$$

and which will be satisfied by three given points, can always be found; because three equations of condition will determine the *ratios* of the four coefficients, or the *values* of the constants in

$$y^2 + dx + ey + f = 0.$$

We shall, in this way, generally determine a parabola passing through the three given points. Thus, let the points be (3, 1), (2, — 2) and (— 1, 5). Assuming the above form, the equations of condition are

$$1 + 3d + e + f = 0,$$
$$4 + 2d - 2e + f = 0,$$
and
$$25 - d + 5e + f = 0,$$

from which, eliminating f, we have

$$3 - d - 3e = 0 \text{ and } 24 - 4d + 4e = 0;$$

and finally, $d = 5\frac{1}{4}$, $e = -\frac{3}{4}$, $f = -16$. The required parabola, therefore, is $y^2 + 5\frac{1}{4}x - \frac{3}{4}y - 16 = 0$, or $4y^2 + 21x - 3y - 64 = 0$, which is satisfied by each of the given points.

If two of the given points have the same ordinate, we shall find $d = 0$, and the equation will represent a pair of lines parallel to the axis of X. If all three ordinates are equal, the equations will be found insufficient to determine e and f; for the points being on one straight line parallel to the axis of X, this and *any* line parallel to it will constitute a pair satisfying the conditions. If the three points are in one straight line, *not* parallel to the axis of X, the equations of condition will be found to be contradictory, and no equation of the form assumed can be found. But, even in this case, an equation of the general form, fulfilling the conditions, can be found; for let $C = 0$, then $Dx + Ey + F = 0$ will with proper values of the coefficients represent the straight line on which the points are situated.

Therefore the general equation of the parabola above becomes the equation of two straight lines when $D = 0$, and becomes that of a single straight line when $C = 0$.*

EXAMPLES.—Determine the equation, for the points (2, — 1), (1, 0) and (3, 2); for (1, 1), (— 1, 5) and the origin; for (1, 2), (0, 5) and (3, — 4); for (2, 0), (2, 3) and (2, 1).

Give the value of the parameter, etc., in each case.

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* When $C = 0$, the equation cannot be put in the form $y^2 + dx + ey + f = 0$; just as when $B = 0$, $Ax + By + C = 0$ cannot be put in the form $y = mx + b$. General expressions for $2p$, x' and y' would all become infinite, for $C = 0$.

Intersections of Parabolas.

160. The intersection of a given straight line with a parabola of the form $Cy^2 + Dx + Ey + F = 0$, is most readily found by substituting the value of x from the straight line in the equation of the parabola. This gives a quadratic equation for y, whose roots are real, equal or imaginary. Accordingly the line is said to cut the curve in two real, coincident or imaginary points; that is, it cuts the curve in two points, touches it, or fails to meet it altogether.

But if the line is parallel to the axis of X; that is, if its equation is of the form $y = b$, this method of solution cannot be used; we have to substitute for y in the parabola its constant value, and that will give an equation of the first degree for x. Thus, given $2y^2 + 3x - 4y + 12 = 0$, and the line $y = 2$, we have $8 + 3x - 8 + 12 = 0$ to determine the value of x, which is $x = -4$. As an equation of first degree gives but a single value of x, such a line *cuts* the curve in a *single* point. This must be distinguished from the case of *two coincident* points, which indicates *tangency.*

Examples.—Find the intersection of $x + y = 2$, with $y^2 + x + 4y + 2 = 0$; of $x + y = 0$, with the same parabola.

Find the intercept of the curve on the axis of X, and its intersections with $x = -2$, and with $y = -2$.

161. The cutting in a single point indicates that the line is a diameter. Hence the form of the equation of the parabola, which has been discussed, shows that the diameters and axis of the curve are parallel to the axis of X. In like manner, an equation of the form

$$Ax^2 + Dx + Ey + F = 0$$

represents a parabola, cut in a single point by every line of the form $x = a$; that is, having its diameters and axis, parallel to the axis of Y. These two forms represent parabolas with particular directions of the axis; the equation of a parabola having its axis in *any* direction, when combined with the equation of a straight line, would generally give two points, real, coincident or imaginary; but for lines in a certain direction, there will be but one point.

It must be remembered that $y^2 + dx + ey + f = 0$ is a parabola already fulfilling one condition, and that in Art. 159 it was shown that besides that condition it may be made to fulfil three others; namely, to pass through three given points. It will be shown in Chap. VIII., that in general a parabola may be made to fulfil *four* conditions, or pass through four given points. One of the conditions in Art. 159 is that the axis have a certain direction.

162. Parabolas whose axes have the same direction may be called *parallel parabolas.* Parabolas evidently may intersect in four points, but *parallel parabolas intersect in only two points.* For, taking the axis of X parallel to the axes of the curves, their equations will be of the forms $y^2 + dx + ey + f = 0$ and $y^2 + d'x + e'y + f' = 0$. The equation formed by combining these is

$$y^2 + dx + ex + f + k(y^2 + d'x + e'y + f) = 0,$$

which, since it contains no term of the second degree, except y^2, generally represents a parabola parallel to the given ones, and passing through all their points of intersection. But when we make $k = -1$; that is, when we combine the equations by simple subtraction, the result is of the first degree and represents a straight line, which can only cut either parabola in *two* points.

This straight line is analogous to the radical axis of two circles, and its equation may be used to find the points of intersection; thus, given $y^2 + 8x - 3y + 10 = 0$ and $y^2 + 4x + 6 = 0$, subtracting, we have $4x - 3y + 4 = 0$. Substituting $4x = 3y - 4$ in $y^2 + 4x + 6 = 0$, $y^2 + 3y + 2 = 0$. This quadratic gives $y = -2$ or $y = -1$, and finding the corresponding values of x from $4x = 3y - 4$, the two points of intersection are $(-2\frac{1}{2}, -2)$ and $(-1\frac{3}{4}, -1)$, These points will be found to satisfy both the given equations.

EXAMPLES.—Find the intersections of $y^2 + 4x - 2y - 18 = 0$ and $y^2 - 4x + 2y = 0$; of $y^2 + 4x + 2y + 6 = 0$, with each of them, and verify the results.

163. If $d = d'$, the given parabolas are equal, as well as parallel, for the parameters are equal, and the signs of the parameters being the same, they extend in the same direction. In this case the equation of the straight line, which is

$$(d - d')x + (e - e')y + f - f' = 0,$$

takes the form $y = b$. Therefore by Art. 160, it can only cut either parabola in one point. Hence, *parallel and equal parabolas extending in the same direction intersect in but one point.*

If at the same time $e = e'$, so that the given equations differ only in their absolute terms, the straight line becomes the line at infinity, and there is *no** intersection, as in the case of concentric circles, and universally, of the loci of equations differing only in the absolute term.

164. Another method of finding the intersections is to combine the equations so as to eliminate x, which gives the equation of a pair of straight lines, as shown in Art. 156. Each of these lines is of the form $y = b$, and intersects either of the given parabolas in a single point. For example, take $y^2 + 8x - 3y + 10 = 0$ and $y^2 + 4x + 6 = 0$, the parabolas whose intersections were found in Art. 162. Subtracting the first from twice the second gives $y^2 + 3y + 2 = 0$.

We thus obtain at once the quadratic which was solved in that Art., giving the two values of y; namely, $y = -2$ and $y = -1$, which are the equations of the two parallel lines in question.

Parabolas Passing through Fixed Points.

165. The equation of combination

$$y^2 + dx + ey + f + k(y^2 + d'x + e'y + f') = 0$$

represents a parabola fulfilling three conditions, and capable of fulfilling a fourth. The first of the three conditions fulfilled is that it is parallel to the given parabolas, because it is included in the general form $Cy^2 + Dx + Ey + F = 0$. The other two are, that it passes through two fixed points—namely, those real, coincident or imaginary points, in which the given parabolas intersect. The fourth condition will determine k; this condition may be that it pass through a third fixed point, as in Art. 159, or that the parabola have a given parameter. The general expression for $2p$ is

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* The intersections which disappear in the above cases are said to be "at infinity." Thus, one of the intersections of a parabola and a line parallel to its axis, or of *equal* parabolas, is at infinity; and when the *axes coincide* both of them are at infinity.

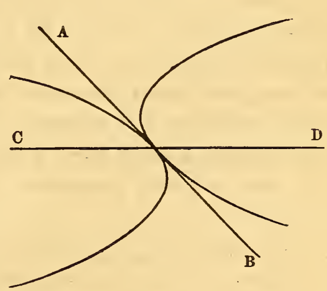
$$2p = -\frac{D}{C} = -\frac{d + kd'}{1 + k},$$

to which, in general, we may give any value and so determine the value of k. Thus, given the parabolas $y^2 + 4x + 6 = 0$ and $y^2 + 8x - 3y + 10 = 0$, and required that the new parabola have a parameter 6, and extend toward the left (that is, $2p = -6$); we have $-6 = -\frac{4 + 8k}{1 + k}$, whence $k = 1$. The required parabola is $2y^2 + 12x - 3y + 16 = 0$.

166. This method of determining k may be regarded as including the cases in which the equation is made to represent a single straight line, and a pair of parallel lines. For $k = -1$ makes the expression for $2p$ infinite, and $k = -\frac{d}{d'}$, makes it zero. Thus, the parabola vanishes into a single straight line, when the parameter is increased without limit. It is easy to see that it *approximates* to a single line when the parameter is very large, just as it approximates to parallel lines when it is very small.

In case $d = d'$, the expression for $2p$ reduces to $-d$, whatever the value of k; that is, the parameter of the parabola represented by the combined equation is constant, when the given parabolas are equal. In Art. 163, it was shown that equal parabolas intersect in only one point, therefore the series of parabolas, on this supposition, are *parallel* and *equal* parabolas passing through *one* fixed point. By an equation of condition we can find one of the series passing also through a given point.

167. Except in the above case, a certain value of k will give a single line, and another a pair of parallel lines. The parallel lines are real, coincident or imaginary, according as the two points common to the system are real, coincident or imaginary. In the figure the parabolas are supposed to touch. The common tangent, AB, is the single line, and CD, parallel to the axes of the curves,



is the pair of coincident lines. Though CD, regarded as a single line, *cuts* the parabolas, yet as a double line it meets them in two coincident points, and therefore fulfils the *algebraic* condition of tangency. In fact, in the equation of any two coincident lines every value of x gives *equal values* of y, therefore it fulfils this condition with respect to every line. The parabola represented by the equation of combination, in this case, fulfils the two conditions of *tangency to a fixed line* at a *fixed point.*

168. If we combine the equation of a pair of lines parallel to the axis of X with that of a single oblique line, thus,

$$y^2 + ey + f + k\,(\mathrm{A}x + \mathrm{B}y + \mathrm{C}) = 0,$$

we still have the equation of a parabola passing through two fixed points, whatever the value of k. The two points are those in which $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ cuts the two lines represented by $y^2 + ey + f = 0$, and k may be determined by an equation of condition, so that the parabola shall pass through a third fixed point. This formula furnishes us another method of finding the parabola passing through three given points. Thus, take the first two of the points in the example solved in Art. 159; namely, (3, 1) and (2 — 2). The lines parallel to the axis of X, passing through these points, are $y - 1 = 0$ and $y + 2 = 0$, the equation of the pair is, by Art. 81, $(y - 1)\,(y + 2) = 0$ or $y^2 + y - 2 = 0$. The equation of the straight line is, by the formula of Art. 64, $y - 1 = 3\,(x - 3)$ or $y - 3x + 8 = 0$, hence

$$y^2 + y - 2 + k\,(y - 3x + 8) = 0,$$

is the equation of a parabola passing through these two points. Now, if the parabola is required to pass through $(-1, 5)$ also, the equation of condition is $28 + 16k = 0$ or $k = -\frac{7}{4}$, hence $y^2 + y - 2 - \frac{7}{4}\,(y - 3x + 8) = 0$ or $4y^2 - 3y + 21x - 64 = 0$, is the required parabola, the same that was found by the other method.

169. In like manner, we may find the general equation of the parabola touching a given line at a given point. Thus, required the parabola touching the line $y = x + 2$, at the point (1, 3), which is a point of the line; we combine with the equation of the

line, the equation of a pair of coincident lines passing through the point. Hence the equation

$$(y-3)^2+k(y-x-2)=0,$$

in which k may be determined by another condition; for instance, that the parabola have a given parameter. In this equation the parameter is k; in the general equation of Art. 168, it is $-kA$.

Of course, all the parabolas found have had axes parallel to the axis of X; but, from the principle of combined equations, the curve would still pass through two fixed points, if the parallel lines had *any* direction; and it will hereafter be proved that it will be a parabola having its axis in that direction.

EXAMPLES.—Solve the examples under Art. 159, by the above method.

Find parabolas with axes parallel to the axis of X and passing through (1, 1) and (1 — 1); 1st, having 8 for parameter; 2d, passing through the origin; 3d, making the intercept $x_0=2$.

Find a parabola touching $x=y$ at (1, 1), and passing through (2, 3); a parabola tangent to the axis of Y at (0, 2), of the same size and direction as $y^2+4x-y=0$.

What is represented by $y^2+dx+ey+f+k(y-b)=0$?

Ans. A series of parallel and equal parabolas passing through a fixed point.

Give the general equations (having one arbitrary constant, k) of the parabola whose parameter is 8, and which passes through (2, 1).

What does $Cy^2+Dx+Ey+F=0$ denote, when D, E and F are fixed, and C is regarded as an arbitrary constant (that is, regarding C as taking the place of k)? what when D only is arbitrary? and what when E only is arbitrary?

EQUATIONS OF THE TANGENT.

170. Since $Y^2=2pX$ represents a parabola, even when the axes of co-ordinates are oblique, the algebraic expressions for the intersections of a straight line and parabola, in Art. 146, apply to a parabola referred to any diameter and the tangent at its vertex. Therefore, also the condition of tangency is the same, and the equation of a tangent and co-ordinates of the point of contact are of the same form—namely,

$$X = nY - \tfrac{1}{2}n^2p,$$

$$X_1 = \tfrac{1}{2}n^2p \qquad \text{and} \qquad Y_1 = np.$$

In these equations p is half the parameter corresponding to the diameter to which the curve is referred, and n determines the direction of the tangent, but is *not* the co-tangent of the inclination, except when the axes are rectangular.

171. In case the axes are rectangular, the equation of the tangent may be expressed in terms of the inclination of a perpendicular upon it from the origin. For, reducing it by the method of Art. 52, to the form $x \cos a + y \sin a - p = 0$, we have

$$\frac{X}{\sqrt{1+n^2}} - \frac{nY}{\sqrt{1+n^2}} + \tfrac{1}{2}p\frac{n^2}{\sqrt{1+n^2}} = 0$$

hence $\quad \cos a = \dfrac{1}{\sqrt{1+n^2}} \quad$ and $\quad \sin a = \dfrac{-n}{\sqrt{1+n^2}}.$

We have now to express the absolute term, $\tfrac{1}{2}p\dfrac{n^2}{\sqrt{1+n^2}}$, in terms of a. From the above values of $\sin a$ and $\cos a$, we find

$$\frac{\sin^2 a}{\cos a} = \frac{n^2}{\sqrt{1+n^2}},$$

hence the equation of the tangent in terms of a, is

$$X \cos a + Y \sin a + \tfrac{1}{2}p\frac{\sin^2 a}{\cos a} = 0.$$

The absolute term, taken negatively, is the perpendicular from the vertex. By the formula of Art. 73, the perpendicular from any point is found by substituting its co-ordinates in the first member; therefore, the perpendicular from the focus, $(\tfrac{1}{2}p, 0)$, is

$$p' = \tfrac{1}{2}p\left(\cos a + \frac{\sin^2 a}{\cos a}\right) = \tfrac{1}{2}p\frac{\cos^2 a + \sin^2 a}{\cos a} = \frac{\tfrac{1}{2}p}{\cos a}.$$

This expression being of the same sign as the absolute term, the focus and vertex are on the same side of any tangent. For a given value of a, the tangent could be constructed by laying off the absolute term negatively from the vertex, or by laying off this value, also negatively, from the focus. Thus, for a value of α in the first quadrant they must be laid off backward, or below the axis, for a value in the second quadrant above the axis, since $\cos \alpha$ is then negative.

EXAMPLES.—Give the values of these perpendiculars for $\alpha = 0°$, 60°, 120°, 180°, etc.

172. If we put the above value of p' in place of the absolute term,* we have the equation of the tangent as referred to the focus,

$$X \cos \alpha + Y \sin \alpha + \frac{\frac{1}{2}p}{\cos \alpha} = 0.$$

The angle α and the value of the perpendicular from the focus or negative of the absolute term, are the polar co-ordinates of the foot of the perpendicular. (See Art. 69.) Denoting them by θ and r, we have the following relation between them,

$$r = -\frac{\frac{1}{2}p}{\cos \theta} \qquad \text{or} \qquad r \cos \theta = -\tfrac{1}{2}p.$$

This is the polar equation of *the locus of the foot of a perpendicular from the focus upon a tangent.* By Art. 71, it is a straight line perpendicular to the axis, at a distance $\frac{1}{2}p$ to the *left* of the focus; that is, *the tangent at the principal vertex.* Therefore we may construct a tangent with a given value of α, by drawing a line in the given direction, and a perpendicular to it at the point where it cuts the vertical tangent, as BF and BP_1, figure Art. 149.

173. The equation of the tangent may be expressed in terms of the co-ordinates of its point of contact, thus: Putting X_1 in place of its value $\frac{1}{2}n^2p$, we have $X = nY - X_1$ or $nY = X + X_1$; then substituting for n its value from $Y_1 = np$, the result is

$$YY_1 = p\,(X + X_1).$$

This is a formula for a tangent at a given point on the parabola.

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* When we *construct* a line from its equation in the form $x \cos \alpha + y \sin \alpha - p = 0$, we have to lay off the *negative* of the absolute term. But, since the formula for p' gives the absolute term itself, as the perpendicular from the origin, we make this transformation by putting the value of p' directly for the absolute term.

Thus, the tangent to $y^2 = 8x$, at the point $(2, 4)$, which we find to be on the curve, is $4y = 4(x + 2)$ or $y = x + 2$.

It must be remembered that this equation, like the corresponding equation for the tangent to a circle, is *not* the equation of a tangent, nor is P_1 a point of the line, *unless* it is also a point of the curve. That is,

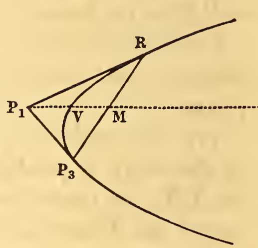
$$Y_1^2 = 2pX_1$$

is the condition of the line's tangency, and this condition also expresses that P_1 is on the line.

174. If we call the straight line represented by this equation the *polar* of the point P_1, with respect to the curve $Y^2 = 2pX$, the equation

$$Y_2Y_1 = p(X_2 + X_1)$$

may be regarded as expressing, either that P_2 is on the polar of P_1, or that P_1 is on the polar of P_2. Therefore such points are said to be *reciprocally polar.* A *self-polar* point, or one on its own polar, is also on the curve, and its polar is a tangent. The problem of finding the equations of tangents from a given point to the curve is, therefore, equivalent to finding the self-polar points (or points of the curve) which are polar to the given point. These points are the intersections of the polar of the given point with the curve. Thus, let the line P_2P_3 be the polar of P_1, then tangents to the curve at P_2 and P_3 will pass through P_1. For example, the polar of $(-1, 2)$, with respect to the parabola $y^2 = 12x$, is $2y = 6(x - 1)$ or $3x = 3 + y$. The intersections of this line with the curve are $(3, 6)$ and $(\frac{1}{3}, -2)$, and the polars of these points, namely, $6y = 6(x + 3)$ or $y = x + 3$, and $-2y = 6(x + \frac{1}{3})$ or $-y = 3x + 1$, are tangents to the curve and pass through the given point $(-1, 2)$.



175. Since in the formula for the polar of P_1,

$$YY_1 = p(X + X_1),$$

the coefficient of X is p, half the parameter, the equation of a polar cannot take the form $y = b$, nor the impossible form $C = 0$; therefore every point has a polar, and the polar cannot be parallel to the axis. The polar of a point on the axis of X, that is, on the diameter to which the curve is referred, found by making $Y_1 = 0$, is $0 = p(X + X_1)$ or $X = -X_1$, which represents a line parallel to the axis of Y, and cutting the diameter in a point at the same distance from the vertex, as the point P_1, but on the opposite side. We can, therefore, construct the polar of P_1, in the figure, by drawing a diameter through it, and then laying off $VM = P_1V$, and drawing a line parallel to the tangent at V. Therefore, P_2P_3 is a double ordinate to this diameter, and the tangents at the extremities of any double ordinate meet in the diameter.

If P_1 were on the right of V, the polar would cut the diameter produced, and would not cut the curve, therefore no tangents could be drawn through a point within the curve. The polar of a point on the axis is perpendicular to it, and the polar of the focus is the directrix. Hence if P_1 is on the directrix, it is reciprocally polar to the focus, or its polar passes through the focus. In that case, P_2P_3 would be a focal chord, and therefore the tangents at the extremities of a focal chord meet on the directrix.

EXAMPLES.—Prove by the equation of the tangent line, that the tangents at the extremities of a double ordinate meet on the diameter which bisects it, and that the intercepted part of the diameter, as P_1M in the figure, is bisected at V.

Prove by considering the equation of a polar as referred to any diameter, that points situated on any other diameter have parallel polars.

176. The general equation of a tangent or polar, when the axis of the curve is parallel to the axis of X, is found by substituting for X, X_1, etc. (which are co-ordinates as referred to a point on the curve), the differences $(x - x')$, $(x_1 - x')$, etc., which denote the same quantities. Hence

$$(y - y')(y_1 - y') = p(x + x_1 - 2x')$$

is the formula for a polar or tangent with respect to

$$(y - y')^2 = 2p(x - x').$$

Expanding it, we have

$$yy_1 - y'(y + y_1) - p(x + x_1) + y'^2 + 2px' = 0.$$

Finally, introducing the constants d, e and f, with the same values as in Art. 154, where the expanded equation of the parabola is simplified, we have

$$yy_1 + \tfrac{1}{2}d\,(x + x_1) + \tfrac{1}{2}e\,(x + y_1) + f = 0,$$

for the tangent or polar with respect to

$$y^2 + dx + ey + f = 0.$$

Therefore, given the equation of a parabola, we find the formula for the polar in the same way as for the equation of the circle, by putting yy_1 in place of y^2, and $\frac{1}{2}(x + x_1)$, $\frac{1}{2}(y + y_1)$ in place of x and y. Thus, for the polar with respect to $y^2 + 2x - 6y + 15 = 0$, the formula is $yy_1 + x + x_1 - 3(y + y_1) + 15 = 0$. Now, suppose we require the equations of tangents to this parabola through the point (1, 2). By the formula, the polar of this point is $2y + x + 1 - 3y - 6 + 15 = 0$ or $x = y - 10$. This line will be found to cut the curve in $(-5, 5)$ and $(-11, -1)$. The polars of these points by the same formula are $2y + x = 5$ and $x - 4y + 7 = 0$, which are tangents to the parabola and pass through the given point (1, 2).

Examples.—Give the polar formula for $y^2 - 4x + 7y + 10 = 0$.

Find the equations of tangents to this curve, passing through $(1, -1)$; through the origin; etc., etc.

Prove, from the general formula, the properties of "reciprocally polar" and "self-polar" points.

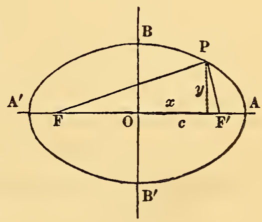
Give the general equation of the polar to the origin, and show that it cuts the curve (and therefore the origin is without the curve), only when f is positive.

CHAPTER VI.

THE ELLIPSE.

177. If a point move in such a manner that the sum of its distances from two fixed points is constant, it will describe a curve called an *ellipse*; the fixed points are called the *foci*.

To find the equation of the ellipse, take for the axis of X the straight line passing through the foci, F and F′, in the figure; and for the origin a point midway between them; and let the axes be rectangular. Denote the distance from either focus to the origin by c, and the constant sum of the distances PF and PF′ by 2A. Let r and r' represent these distances, or the *focal distances* of the point P; then, by right triangles,



$$r^2 = y^2 + (c + x)^2 \qquad \text{and} \qquad r'^2 = y^2 + (c - x)^2.$$

These are the geometrical relations between the focal distances and co-ordinates of P, and we have, besides, the equation,

$$r + r' = 2\text{A},$$

expressing the definition of the curve. Between the three equations, we have now to eliminate the variables, r and r', so as to obtain an equation between x, y and the constants. To do this we must find expressions for r and r' in terms of x and y. To avoid radicals, divide the difference of the squares $r^2 - r'^2 = 4cx$, by the sum $r + r' = 2\text{A}$, which gives the difference $r - r' = \dfrac{2cx}{\text{A}}$. Combining this with $r + r' = 2\text{A}$,

$$r = \text{A} + \frac{cx}{\text{A}} \qquad \text{and} \qquad r' = \text{A} - \frac{cx}{\text{A}}.$$

Substituting the value of r in the first equation, or of r' in the second, the middle terms of the expanded squares disappear, and we have

$$A^2 + \frac{c^2x^2}{A^2} = y^2 + c^2 + x^2 \quad \text{or} \quad y^2 + \frac{A^2 - c^2}{A^2}x^2 = A^2 - c^2.$$

178. This is the equation of the ellipse in terms of the constants A and c. It will take a more convenient form, if we introduce another constant in place of $A^2 - c^2$. Since A is necessarily greater than c, this quantity is always positive, and may therefore be denoted by B^2; making the substitution and dividing by B^2,

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1.$$

Now, if we make $y = 0$, we have $x^2_0 = A^2$, $x_0 = \pm A$; that is, the curve cuts the axis of X at the distance A, to the right and left of the origin. In like manner, $x = 0$ gives $y_0 = \pm B$, or the curve cuts the axis of Y at the distance B, above and below the origin. The parts of the axes A′A and B′B intercepted by the curve are called the *major* and the *minor* axis, and OA, OB whose lengths are A and B are the *semi-axes.* The point O, in which the axes meet, is called the *centre* of the ellipse; and the points A, B, A′ and B′, *vertices* of the major and the minor axis. The ratio $\frac{c}{A}$, or distance of the focus from the centre divided by the major semi-axis, is called the *eccentricity* of the ellipse, and is denoted by e. Using this constant, the expressions for the focal distances of the point become

$$r = A + ex \qquad \text{and} \qquad r' = A - ex.$$

The focal distances of the vertex A, whose abscissa is A, are $A + c$ and $A - c$; the focal distances of B, whose abscissa is zero, are each equal to A.

179. Since the centre is the point to which the ellipse is most readily referred, we shall use X and Y, in the manner explained in Art. 96, as central co-ordinates; then (clearing of fractions),

$$A^2Y^2 + B^2X^2 = A^2B^2$$

is the *central equation* of the ellipse, which we shall use in investigating its form, and

$$A^2 (y - y')^2 + B^2 (x - x')^2 = A^2B^2$$

is the equation of the ellipse with centre at P′, and axes parallel to the axes of co-ordinates.

In finding the central equation we took the major axis (that on which the foci are situated) as the axis of X, therefore A in the equation is supposed greater than B, and the foci are the points $(-c, 0)$, $(c, 0)$, in which c is determined by

$$c^2 = A^2 - B^2.$$

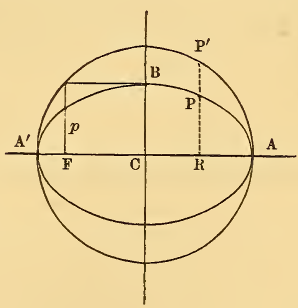
If $A = B$, $c = 0$, and both foci coincide with the centre; in which case, the point P is at a constant distance from the centre, and describes a circle: accordingly we find that under this supposition, the equations represent circles; that is, the ellipse with equal axes is a circle.

Form of the Ellipse.

180. The central equation may be put in the form,

$$Y^2 = \frac{B^2}{A^2} (A + X) (A - X).$$

Draw the ordinate PR, of any point; then $A + X$ and $A - X$ are the segments A′R and AR, into which it divides the major axis, and the equation expresses that "the square of a perpendicular has a constant ratio to the product of the segments into which it divides the axis." Construct a circle on the major axis as diameter. In Art. 106, we saw that the square of the ordinate P′R of the circle is equal to the product of these segments; therefore $Y^2 : P'R^2 :: B^2 : A^2$, or the ordinates of the ellipse and this circle, *corresponding to the same abscissa*, are in the constant ratio, B to



A. This ratio (of the minor to the major axis) therefore determines the *shape* of the ellipse, while the length of the major axis determines its *size*. There are thus *two* respects in which ellipses differ from one another, while circles and parabolas differ only in one respect, that of size.

In a similar manner, we may show that the abscissas of the ellipse and the circle on the minor axis, *corresponding to the same ordinate*, are in the constant ratio A : B, and the property, above proved, of the square of a perpendicular, is true for both axes. Hence, *if the ordinates of a circle be all increased or reduced in a given ratio, the resulting curve will be an ellipse;* or if the point P, in the figure, move in a perpendicular to the diameter of the circle, so that PR is in a constant ratio to P′R, it will describe an ellipse.

181. The ordinate corresponding to either focus is found by making $X = \pm c$, or $X^2 = c^2$, which gives $A^2Y^2 = B^4$, since $A^2 - c^2 = B^2$; hence $Y = \frac{B^2}{A}$, or the ordinate required is a third proportional to the major and minor semi-axes. The double ordinate passing through the focus is called the *parameter*, and is denoted by $2p$, as in the parabola; since $2A : 2B :: 2B : 2p$, it is a third proportional to the axes. The product of the segments into which the ordinate p divides the axis is $(A - c)\ (A + c) = A^2 - c^2 = B^2$; hence the corresponding ordinate in the circle equals B. The figure indicates a method of finding a focus, from this property, which is evidently equivalent to making BF = AC. This construction also shows that $p : B :: B : A$, and that B is a geometrical mean between A′F and AF.

Polar Equations of the Ellipse.

182. By the formulæ of transformation to polar co-ordinates, the central equation of the ellipse becomes

$$r^2 = \frac{A^2B^2}{A^2 \sin^2 \theta + B^2 \cos^2 \theta}.$$

This equation gives equal positive and negative values of r for each value of θ; therefore every chord passing through the centre is there bisected. Such a chord is called a *diameter*. Putting for B^2 in the denominator its value $A^2 - c^2$, we have

$$r^2 = \frac{A^2B^2}{A^2 - c^2 \cos^2 \theta} = \frac{B^2}{1 - e^2 \cos^2 \theta}.$$

These are expressions for the square of the semi-diameter, whose inclination is θ. When $\theta = 0°$, we find $r^2 = A^2$, which is its greatest possible value, and when $\theta = 90°$, $r^2 = B^2$, its least possible value. All values of r, or semi-diameters, are therefore intermediate in length between A and B, the semi-axes.

EXAMPLES.—The semi-axes being 5 and 3, find the semi-diameter inclined 30° to the major axis; that inclined 45°; that inclined 60°.

Prove that the sum of the squares of reciprocals of perpendicular semi-diameters is constant.

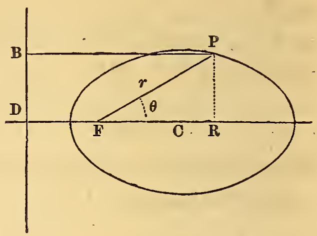
183. The polar equation, when the pole is at the focus F, or left-hand focus, may be found thus: In the value of r, Art. 178, x is the abscissa of P as measured from the centre C, or CR, in the next figure; but in polar co-ordinates the value of CR is evidently $r \cos \theta - c$. Hence $r = A + e\,(r \cos \theta - c)$. The value of r takes its simplest form when expressed in terms of e and p, the eccentricity and parameter. Between the constants A, B, c, e and p, we have by combining $B^2 = A^2 - c^2$, $e = \frac{c}{A}$ and $p = \frac{B^2}{A}$, the following relations:

$$c = Ae, \qquad B^2 = A^2\,(1 - e^2), \qquad p = \frac{B^2}{A} = A\,(1 - e^2).$$

Making substitution of these values, the above equation becomes $r = A + er \cos \theta - Ae^2 = p + er \cos \theta$; hence

$$r = \frac{p}{1 - e \cos \theta}.$$

In this equation e determines the shape of the ellipse, and p determines its size. If we suppose $e = 1$, the value of r reduces to that which we found in Art. 143, for the parabola. Hence the ellipse becomes a parabola when the eccentricity becomes unity. If we make $e = 0$, it reduces to $r = p$, the equation of a circle whose radius is the semi-parameter. Hence the circle is an ellipse with *no* eccentricity. For the ellipse proper, the value of e is a proper fraction, because c is less than A; that is, the eccentricity is between the limits one and zero, for which the ellipse becomes a parabola and a circle.



184. The polar equation, just found, is equivalent to a very simple relation between the radius vector and the abscissa of P. For $r = p + er\cos\theta$, or

$$r = p + ex.$$

Here x is FR, the abscissa measured from the focus, and p is evidently the radius vector corresponding to $x = 0$. If we find a point D, on the left of F, at the distance $\text{FD} = \frac{p}{e}$, then $p = e\,\text{FD}$,

and $$r = e\,(\text{FD} + x) = e\text{DR}.$$

Draw the line DB perpendicular to the axis at D, then $\text{DR} = \text{PB}$, the perpendicular distance of P from this line, and we have proved that *the distance of* P, *any point of the ellipse, from the fixed point* F, *and the fixed line* DB, *are in the constant ratio* $e : 1$. The fixed line is called the *directrix*. Since $e < 1$, the vertex is nearer F than D, and from the value of p, in the last Article, we find the distance of the centre from this vertex, $\text{A} = \frac{p}{1 - e^2}$.

When $e = 1$, the vertex is midway between F and D, and the centre is at an infinite distance, the curve becoming a parabola. When $e = 0$, FD becomes infinite, therefore the circle has no directrix.

Examples.—Determine the parameter and eccentricity of the ellipse whose semi-axes are 5 and 3, and give its polar equation.

What values of r correspond to $\theta = 0°$? $\theta = 60°$? $\theta = 90°$?

Determine the greatest and the least value of r in terms of p and e, and show, by the relations between the constants, that they are equivalent to $\text{A} + c$ and $\text{A} - c$.

What value of θ makes $r = \text{A}$? *Ans.* $\cos\theta = e$.

185. The value of r given by the polar equation is always positive, because e and $\cos\theta$ are less than 1. There is also a negative value of r corresponding to each value of θ. Both values may be

found by transformation from the rectangular equation. The focus being the origin, the co-ordinates of the centre are c and 0. Substituting these values for x' and y', the co-ordinates of the centre, in $A^2(y-y')^2 + B^2(x-x')^2 = A^2B^2$, we have $A^2y^2 + B^2x^2 - 2B^2cx = A^2B^2 - B^2c^2 = B^4$. Transforming to polar co-ordinates,

$$r^2(A^2 \sin^2\theta + B^2\cos^2\theta) - 2B^2cr\cos\theta = B^4.$$

If we substitute for B^2 its value $A^2(1-e^2)$, Art. 183, the coefficient of r^2 becomes $A^2(1-e^2\cos^2\theta)$; then dividing through by A^2, and putting p and e for their values $\frac{B^2}{A}$ and $\frac{c}{A}$,

$$r^2(1-e^2\cos^2\theta) - 2per\cos\theta = p^2,$$

hence $$r^2 = p^2 + 2per\cos\theta + e^2r^2\cos^2\theta,$$

and $$r = \pm(p + er\cos\theta).$$

The upper sign gives the value of Art. 183, the lower gives

$$r = \frac{-p}{1+e\cos\theta},$$

which is always negative.

This value of r, taken with the positive sign, is the same that would be found for the right-hand focus. It is also the same that we obtain for $180° + \theta$. The arithmetical sum of the two values gives the length of a focal chord,

$$\frac{2p}{1-e^2\cos^2\theta}.$$

When $e=1$, this value reduces to the expression we found for the parabola, Art. 145.

186. Placing the centre at the point $(A, 0)$, so that the origin is the left-hand vertex of the major axis, the rectangular equation is $A^2y^2 + B^2x^2 - 2AB^2x = 0$, and transforming,

$$r^2(A^2\sin^2\theta + B^2\cos^2\theta) = 2AB^2r\cos\theta.$$

Making the same substitutions of constants as above, etc., we have

$$r^2(1-e^2\cos^2\theta) = 2pr\cos\theta,$$

which is, of course, always satisfied by $r=0$, because the pole is a point of the curve. The other value of r is

$$r = \frac{2p \cos \theta}{1 - e^2 \cos^2 \theta}.$$

If $e = 1$, this reduces to the corresponding equation for the parabola, Art. 142, and if $e = 0$, to $r = 2p \cos \theta$, the equation of a circle whose radius is p, Art. 115.

EXAMPLES.—Find the polar equation when the pole is the lower vertex of the minor axis.

Find the polar equation when the centre is at the point (c, p); give the values of r corresponding to $\theta = 0$; $\theta = 90$; and that of θ corresponding to $r = 0$.

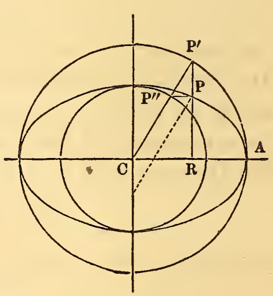
ECCENTRIC ANGLE.

187. It is frequently desirable to express the two co-ordinates of a point of the ellipse, by means of a single variable. In the central equation

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1,$$

since $X < A$ and $Y < B$, there must be an angle whose cosine is $\frac{X}{A}$ and whose sine is $\frac{Y}{B}$; because these quantities are proper fractions, and the sum of their squares is, by the equation, unity. Denoting this angle by φ, we have

$$X = A \cos \varphi, \qquad Y = B \sin \varphi.$$



φ is here a *variable* angle, of which X and Y are such functions that they necessarily satisfy the equation; because, whatever the value of φ, $\cos^2 \varphi + \sin^2 \varphi = 1$. Take any point P on the ellipse; the corresponding value of φ may be constructed thus: Prolong the ordinate of P until it meets the circle constructed on the major axis in P′; draw the radius P′C, then P′CA, its inclination to the major axis, is the angle φ. For by the definitions, $\frac{CR}{CP'}$, or $\frac{X}{A} = \cos P'CA$; and $\frac{Y}{B} = \frac{P'R}{CP'} = \sin P'CA$, since by Art. 180 the ordinates in the ellipse and circle are in the ratio B : A.

In the circle, φ is the same as θ, the angular co-ordinate of the point, when the pole is at the centre. In the ellipse it is called the *eccentric angle* of the point.

EXAMPLES.—Give the eccentric angles of the extremities of each axis.

What is the eccentric angle of the extremity of the parameter?

Ans. The angle whose cosine is e, and whose sine is $\frac{B}{A}$.

In the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$? what are the co-ordinates of the point for which $\varphi = 60°$? of the point for which $\varphi = 30°$? etc.

188. If we construct a circle on the minor axis as diameter, the ordinate of the point P″, in which P′C cuts this circle, will be $B \sin \varphi$, which is also the ordinate of P; hence PP″ is parallel to the major axis. From these properties it is easy to prove that if a line be drawn through P, parallel to P′C, it will cut the major and minor axes at distances from P respectively equal to B and A.

SECANT AND TANGENT LINES.

189. In finding general expressions for the intersection of a straight line and ellipse, we shall use the form

$$Y = mX + b,$$

for the straight line, and the central equation of the ellipse. X and Y are, therefore, co-ordinates as measured from the centre, b is the intercept on the minor axis, and, since the axes are rectangular, m is the tangent of the line's inclination to the major axis. Substituting the value of Y in

$$A^2Y^2 + B^2X^2 = A^2B^2,$$

and dividing by the coefficient of X^2, we have the quadratic

$$X^2 + 2\frac{A^2mb}{A^2m^2 + B^2}X = \frac{A^2B^2 - A^2b^2}{A^2m^2 + B^2},$$

completing the square, etc.,

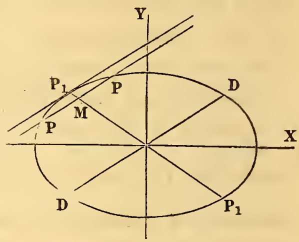
$$X = \frac{-A^2mb \pm AB\sqrt{A^2m^2 + B^2 - b^2}}{A^2m^2 + B^2},$$

and $$Y = \frac{B^2b \pm m\,AB\sqrt{A^2m^2 + B^2 - b^2}}{A^2m^2 + B^2},$$

the values of Y being derived from $Y = mX + b$.

When the radical $\sqrt{A^2m^2 + B^2 - b^2}$ is real, these are the co-ordinates of the points P, P, in which the straight line cuts the ellipse; and the rational parts

$$-\frac{A^2mb}{A^2m^2 + B^2} \quad \text{and} \quad \frac{B^2b}{A^2m^2 + B^2}$$



are the co-ordinates of M, the middle point of the chord PP. Suppose now the value of m to remain fixed while that of b varies, both co-ordinates of M will vary, but their ratio, which is $\frac{Y}{X} = -\frac{B^2}{mA^2}$, is constant. Therefore the middle points of all chords parallel to PP, in the figure, are situated on the line MC, passing through the centre, whose equation is $Y = m'X$, where m' stands for the constant ratio $-\frac{B^2}{mA^2}$. Hence *a system of parallel chords of the ellipse is bisected by a diameter.*

190. If the radical part of the values of X and Y is zero, the line is a tangent. Therefore the condition of tangency is $b^2 = A^2m^2 + B^2$ or $b = \pm\sqrt{A^2m^2 \times B^2}$. Thus, if $A = 2$ and $B = 3$, and m, the direction ratio is required to be 2, then $b = \pm 5$ will make the line a tangent; and $y = 2x + 5$, $y = 2x - 5$ are two tangents having the same direction ratio. In general, substituting the algebraic value of b, we have

$$Y = mX \pm \sqrt{A^2m^2 + B^2},$$

for the equations (as referred to the centre of the ellipse) of two parallel tangents. Let P_1 denote the point of contact of a tangent, then substituting the above values of b in the co-ordinates of M (since for a tangent P,P and M coincide),

$$X_1 = \frac{-A^2m}{\pm\sqrt{A^2m^2 + B^2}} \qquad \text{and} \qquad Y_1 = \frac{B^2}{\pm\sqrt{A^2m^2 + B^2}}.$$

In the figure, m is positive, and X_1 and Y_1 are of opposite signs, the positive value of Y_1 and negative of X_1 belong to the tangent for which the radical or value of b is positive, the others, to that in which b is negative. The points of tangency are the extremities of the diameter which bisects chords parallel to the tangents; they are sometimes called *vertices* of this diameter, the extremities of the axes being distinguished as *principal* vertices.

191. Draw the diameter DD parallel to the tangents and chords; its equation is $Y = mX$. The equation of the diameter P_1P_1 we found to be $Y = m'X$, in which $m' = -\frac{B^2}{mA^2}$, therefore these two diameters are connected by the relation

$$mm' = -\frac{B^2}{A^2}$$

between their direction ratios. When this relation exists, one of the diameters bisects chords parallel to the other. Thus, if $B = 1$ and $A = 2$, $mm' = -\frac{1}{4}$, and the diameter $Y = \frac{1}{8}X$ bisects chords parallel to $Y = -2X$. But the latter must also bisect chords parallel to the former, and in general,

$$Y = mX \qquad \text{and} \qquad Y = m'X,$$

are *each* parallel to chords bisected by the other, or to tangents touching the ellipse at the vertices of the other. Such diameters are called *conjugate diameters* of the ellipse. The axes themselves are a pair of conjugate diameters.

EXAMPLES.—Find tangents to the ellipse whose semi-axes are 5 and 3: 1st, parallel; 2d, perpendicular to $y = 2x - 4$; and find their points of contact.

Find the diameter conjugate to $y = 2x$, and the tangents at the vertices of $y = 2x$.

Show that (X_1, Y_1) satisfies the equation of the ellipse.

EQUATIONS OF THE TANGENT.

192. The equation of a tangent to the ellipse at a given point may be expressed in terms of φ, the eccentric angle of the point. For this purpose take the tangent

$$Y = mX + \sqrt{A^2m^2 + B^2},$$

which touches the ellipse in $X_1 = \dfrac{-A^2m}{\sqrt{A^2m^2 + B^2}}$, $Y_1 = \dfrac{B^2}{\sqrt{A^2m^2 + B^2}}$. Since φ is the eccentric angle of P_1, $X_1 = A\cos\varphi$ and $Y_1 = B\sin\varphi$: comparing the values of X_1 and of Y_1, we can prove

$$\sqrt{A^2m^2 + B^2} = \frac{B}{\sin\varphi}, \qquad \text{and} \qquad m = -\frac{B\cos\varphi}{A\sin\varphi}.$$

Substituting these values and clearing of fractions, we have

$$A\sin\varphi .\ Y + B\cos\varphi .\ X = AB,$$

in which the arbitrary constant m is replaced by φ.

EXAMPLES.—Give the equations of tangents at each of the principal vertices; at the points for which $\varphi = 60°$, $\varphi = 45°$; etc.

Prove that a line touching the ellipse at the extremity of the parameter meets the major axis produced at D (Fig. Art. 184), and makes an intercept on the minor, equal to the semi-major axis.

Verify that the point, whose eccentric angle is φ, satisfies the equation of the tangent.

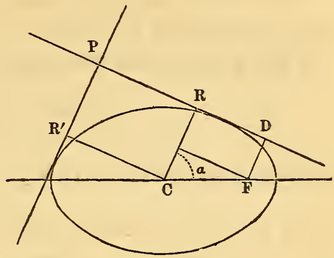
193. The equation of the tangent may also be expressed in terms of the inclination of a perpendicular. For this purpose we must reduce the equation to the form $x\cos\alpha + y\sin\alpha = p$, and then express the value of p in terms of α. Thus (Art. 52),

$$\frac{Y}{\sqrt{1+m^2}} - \frac{mX}{\sqrt{1+m^2}} = \sqrt{\frac{A^2m^2 + B^2}{1+m^2}},$$

hence $\cos\alpha = \dfrac{-m}{\sqrt{1+m^2}}$ and $\sin\alpha = \dfrac{1}{\sqrt{1+m^2}}$.

From these values we see that the quantity under the radical sign, or square of the perpendicular from the centre, is $A^2\cos^2\alpha + B^2\sin^2\alpha$. Hence

$$X \cos a + Y \sin a = \sqrt{A^2 \cos^2 a + B^2 \sin^2 a}$$



is the equation of a tangent, in which a is the arbitrary constant. We give the positive sign to the second member, so that a shall be the inclination of the perpendicular itself, and not of the perpendicular produced. The equation of the tangent parallel to PR is produced by replacing a by $a + 180°$, from which it is evident that perpendiculars on parallel tangents are equal.

Examples.—Give the equation of the tangent when $a = 0°$; when $a = 45°$; when $a = 90°$; when $a = 180°$; etc.

194. The following properties of the ellipse may be demonstrated by aid of the preceding equation.

The sum of the squares of the perpendicular falling from the centre upon perpendicular tangents, is constant, and equals $A^2 + B^2$. For the square of CR, whose inclination is a, is

$$CR^2 = A^2 \cos^2 a + B^2 \sin^2 a,$$

and the square of CR′, perpendicular to the tangent PR′, is

$$CR'^2 = A^2 \sin^2 a + B^2 \cos^2 a,$$

because its inclination is $90° + a$, and $\cos(90° + a) = -\sin a$, $\sin(90° + a) = \cos a$. This last expression is the square of the perpendicular on the tangent whose direction is a. Adding the expressions, $CR^2 + CR'^2 = A^2 + B^2$.

Since CRPR′ is a rectangle, the sum of the squares of two adjacent sides is equal to the square of the diagonal. Therefore $CP^2 = A^2 + B^2$; that is, the intersection of perpendicular tangents is at a constant distance from the centre; and its locus is a circle, whose radius is the distance between the vertices of the axes.

The product of the perpendiculars from the foci upon a tangent is constant, and equals B^2. For the perpendiculars from the foci $(c, 0)$ and $(-c, 0)$ are by Art. 73,

$$c \cos \alpha - \sqrt{A^2 \cos^2 \alpha + B^2 \sin^2 \alpha}$$

and $$- c \cos \alpha - \sqrt{A^2 \cos^2 \alpha + B^2 \sin^2 \alpha},$$

the product of which is B^2.

A perpendicular from a focus meets a tangent in a point of the circle described on the major axis. For RD equals $c \sin \alpha$. Hence

$$CD^2 = CR^2 + RD^2 = A^2 \cos^2 \alpha + (B^2 + c^2) \sin^2 \alpha = A^2,$$

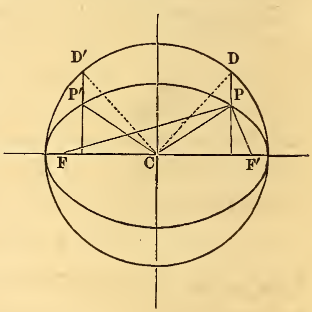
or $CD = A$. That is, the locus of the point D is the circle whose centre is C, and radius equals A.

Conjugate Diameters.

195. In Art. 191 it was shown that when

$$mm' = -\frac{B^2}{A^2}$$

each of the diameters of the ellipse whose direction ratios are m and m', bisects chords parallel to the other, and is parallel to tangents at the vertices of the other; and such diameters were called *conjugate.* Since the axes are rectangular, m and m' are the trigonometric tangents of the inclinations of the diameters, and the relation between their directions is that the product of the tangents of their inclinations is negative, and equal to the square of the ratio of the semi-axes.



Let CP and CP' be a pair of conjugate semi-diameters, and let φ and φ' denote the eccentric angles of their vertices P and P'. The diameter $Y = mX$ passes through P, whose co-ordinates are $A \cos \varphi$ and $B \sin \varphi$, hence $m = \frac{Y}{X} = \frac{B \sin \varphi}{A \cos \varphi} = \frac{B}{A} \tan \varphi$.* Substituting values of m and m' in terms of φ and φ', we have the relation

$$\tan \varphi, \tan \varphi' = -1, \quad \text{or} \quad \tan \varphi' = -\cot \varphi.$$

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* In Art. 192, m denotes the direction ratio of the tangent at the point whose eccentric angle is ϕ, and is equivalent to m' of this Article.

Therefore the radii of the circle constructed on the major axis, CD and CD′, whose inclinations are ψ and ψ' (Art. 187), are perpendicular. Hence *the eccentric angles of the vertices of conjugate diameters differ by* 90°.

196. Adding the squares of the co-ordinates of P, we shall have the square of the semi-diameter CP; hence

$$CP^2 = A^2 \cos^2 \varphi + B^2 \sin^2 \varphi.$$

In like manner, $CP'^2 = A^2 \cos^2 \varphi' + B^2 \sin^2 \varphi'$, or since $\varphi' = \varphi \pm 90°$,

$$CP'^2 = A^2 \sin^2 \varphi + B^2 \cos^2 \varphi.$$

Adding, $$CP^2 + CP'^2 = A^2 + B^2.$$

Hence *the sum of the squares of conjugate semi-diameters is constant, and equal to* $A^2 + B^2$.

Since in the circle the eccentric angle of a point is the same as the inclination of the radius, conjugate diameters of that curve are perpendicular; thus, CD and CD′ are conjugate. Accordingly we found in Art. 119, that "a perpendicular from the centre bisects a chord." Every pair of conjugate diameters in the circle is equal, as well as perpendicular; but in the ellipse, there is but one equal pair, and one perpendicular pair. The latter are the axes, the greatest and least diameters; but in the circle any pair of perpendicular diameters may be taken as the axes.

Examples.—In the ellipse whose semi-axes are 7 and 1, what is the length of the semi-diameter for whose vertex $\varphi = 30°$? of its conjugate? of the equal conjugate pair?

Find in general, the semi-diameters for $\varphi = 0°$, $\varphi = 90°$, and find the length and eccentric angles of the equal conjugate pair.

197. Draw the focal distances PF, PF′. By Art. 178 their values are $r = A + ex$ and $r' = A - ex$. Therefore

$$rr' = A^2 - e^2x^2.$$

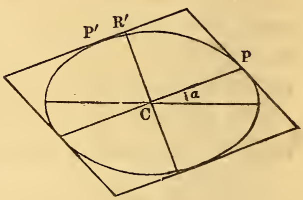
Now, in the value of CP'^2, if we substitute for B^2 its value $A^2 - c^2$, we have $CP'^2 = A^2 - c^2 \cos^2 \varphi = A^2 - A^2e^2 \cos^2 \varphi$, or since φ is the eccentric angle of P, whose abscissa is denoted above by x,

$$CP'^2 = A^2 - e^2x^2 = rr';$$

that is, *the product of the focal distances of the vertex of a diameter equals the square of the conjugate semi-diameter.*

It follows that the square of a semi-diameter, added to the product of the focal distances of its vertex, equals $A^2 + B^2$.

198. If through the vertices of a pair of conjugate diameters tangents to the ellipse be drawn, they will form a parallelogram whose sides are equal and parallel to the conjugate diameters. The area of a parallelogram is equal to the product of one of two opposite sides by the perpendicular distance between them. Therefore the proposed parallelogram is the product of a diameter by the distance between tangents parallel to it, or four times the product of a semi-diameter by the perpendicular from the centre upon a parallel tangent. Let a represent the direction of the tangent; then by the value of CR'^2, Art. 194, the square of the perpendicular upon the tangent is



$$CR'^2 = A^2 \sin^2 a + B^2 \cos^2 a,$$

and by Art. 182, the square of CP, the semi-diameter whose direction is a, is

$$CP^2 = \frac{A^2B^2}{A^2 \sin^2 a + B^2 \cos^2 a};$$

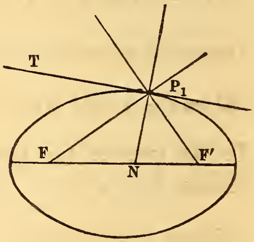
multiplying, etc., we have

$$4CP \times CR' = 4AB = 2A \times 2B;$$

that is, *the parallelogram formed by tangents at the vertices of any conjugate diameters is equal to the rectangle of the axes.*

Lines Bisecting the Angles of Focal Lines.

199. For the equations of the lines joining any point of the ellipse to the foci, we make use of the formula for a straight line passing through two known points,



$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x').$$

Let φ be the eccentric angle of the point P_1 on the ellipse, then the co-ordinates of P_1 are $(A \cos \varphi, B \sin \varphi)$. Substituting these for x'' and y'', and for x' and y', the co-ordinates of F, $(-c, 0)$, and clearing of fractions, we have the equation of P_1F,

$$(A \cos \varphi + c)\, y = B \sin \varphi\, (x + c).$$

In like manner, we find the equation of P_1F', passing through the focus $(c, 0)$,

$$(A \cos \varphi - c)\, y = B \sin \varphi\, (x - c).$$

We have here the equations of P_1F and P_1F' expressed in terms of the constants of the ellipse and the single arbitrary constant φ.

200. To find the equations of the lines bisecting the angles between these lines, we make use of the method of Art. 78; that is, we multiply the terms of each equation, throughout, by the square root of the sum of the squares of the coefficients of x and y, *in the other;* and then add and subtract. In the first place, to find these radicals: The sums of the squares of the coefficients are

$$A^2 \cos^2 \varphi \pm 2Ac \cos \varphi + c^2 + (A^2 - c^2) \sin^2 \varphi,$$

since $B^2 = A^2 - c^2$. (The upper sign belongs to the first equation, the lower to the second.) These expressions reduce to

$$A^2 \pm 2Ac \cos \varphi + c^2 \cos^2 \varphi = (A \pm c \cos \varphi)^2,$$

therefore the radical for the first equation is $A + c \cos \varphi$, and for the second it is $A - c \cos \varphi$,

We therefore multiply both members of the equation of P_1F by $A - c \cos \varphi$. The coefficient of y in the first member becomes

$$(A^2 - c^2) \cos \varphi + Ac\, (1 - \cos^2 \varphi), \quad \text{or} \quad B^2 \cos \varphi + Ac \sin^2 \varphi.$$

Hence the equation reduces to

$$(B^2 \cos \phi + Ac \sin^2 \phi)\, y = (A - c \cos \phi)\, B \sin \phi.\, x + (A - c \cos \phi)\, Bc \sin \phi.$$

In like manner, the equation of P_1F', multiplied by $A + c \cos \varphi$, reduces to

$$(B^2 \cos \phi - Ac \sin^2 \phi)\, y = (A + c \cos \phi)\, B \sin \phi.\, x - (A + c \cos \phi)\, Bc \sin \phi.$$

Finally, adding the equations of P_1F and P_1F', thus prepared, and dividing each term of the result by 2B, we have the equation of P_1N,*

$$B \cos \varphi . y = A \sin \varphi . x - c^2 \sin \varphi \cos \varphi .$$

Subtracting, and dividing by $2c \sin \varphi$, we have the equation of P_1T.

$$A \sin \varphi . y = - B \cos \varphi . x + AB.$$

201. The last equation is identical with the equation of the tangent, found in Art. 192; therefore P_1T is the tangent to the ellipse at the point P_1, whose eccentric angle is φ. A line perpendicular to a tangent at the point of contact is called a *normal.* Hence, *a tangent bisects the exterior angle of the focal lines, and a normal bisects the interior angle.*

It is easy to show that the point P_1 of the ellipse satisfies both the equations; and that the lines they represent are perpendicular, by Art. 48.

The above equation of the normal is expressed in terms of a single arbitrary constant, φ, the eccentric angle of the point P_1. It may thus be expressed in terms of the co-ordinates of P_1: Multiply both members by AB; then, since $A \cos \varphi = x_1$ and $B \sin \varphi = y_1$, we have

$$B^2 x_1 y = A^2 y_1 x - c^2 x_1 y_1,$$

which is the normal at a given point of the curve.

EXAMPLES.—Find the normals to $A^2y^2 + B^2x^2 = A^2B^2$, at the principal vertices, and at the extremity of the parameter (c, p), expressing the latter in terms of A and e.

Show that all the normals of a circle pass through the centre.

Show that the least distance from its vertex, in which a normal can cut the major axis, and the greatest distance from its vertex, in which it can cut the minor axis, are third proportionals to the semi-axes.

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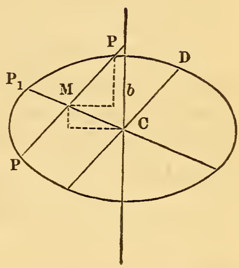
* The absolute terms of the equations were of *contrary* signs, for $A - c \cos \phi$ and $A + c \cos \phi$ are always positive; they are in fact the values of the focal distances P_1F', and P_1F. Therefore *adding* the equations gives the line bisecting the angle in which the origin is situated. See Art. 77.

Ellipse Referred to Conjugate Diameters.

202. Since a diameter bisects chords parallel to its conjugate, the equation of the ellipse referred to conjugate diameters will express the relation between the semi-chords and the parts of the diameter cut off. To find this relation, we resume the values found in Art. 189, for the co-ordinates of the points P,P, in which a straight line cuts the ellipse, namely,

$$X = \frac{-A^2 mb \pm AB\sqrt{A^2m^2 + B^2 - b^2}}{A^2m^2 + B^2},$$

$$Y = \frac{B^2b \pm mAB\sqrt{A^2m^2 + B^2 - b^2}}{A^2m^2 + B^2}.$$



The rational parts of these values are the co-ordinates of M, and the radical parts are the differences of the co-ordinates of M and P. Therefore CM^2 is the sum of the squares of the rational parts, and PM^2 is the sum of the squares of the radical parts. Let CP_1 be taken as the the axis of X, and CD as the axis of Y; and let X and Y represent CM and PM, the oblique co-ordinates of P, then

$$X^2 = \frac{b^2(A^4m^2 + B^4)}{(A^2m^2 + B^2)^2}, \qquad Y^2 = \frac{(1+m^2)A^2B^2(A^2m^2 + B^2 - b^2)}{(A^2m^2 + B^2)^2}.$$

Here m is constant, and b^2 is a variable, upon which the values of X^2 and Y^2 depend. We have, therefore, two equations between three variables, from which to eliminate b^2, and obtain an equation between X^2 and Y^2. The result will be simplified by introducing new constants; namely, the values of the semi-diameters CP_1 and CD. Denote the first by A and the second by B. To find A^2, make $b^2 = A^2m^2 + B^2$ in the value of X^2; because this value of b^2 makes $Y = 0$, and therefore gives the oblique abscissa corresponding to the ordinate zero. To find B^2 put $b = 0$ in the value of Y^2, because $b = 0$ makes $X = 0$. Thus,

$$A^2 = \frac{\mathrm{A}^4m^2 + \mathrm{B}^4}{\mathrm{A}^2m^2 + \mathrm{B}^2} \quad \text{and} \quad B^2 = \frac{(1 + m^2)\,\mathrm{A}^2\mathrm{B}^2}{\mathrm{A}^2m^2 + \mathrm{B}^2}.$$

Dividing X^2 and Y^2 respectively by A^2 and B^2 we derive

$$\frac{X^2}{A^2} = \frac{b^2}{\mathrm{A}^2m^2 + \mathrm{B}^2} \quad \text{and} \quad \frac{Y^2}{B^2} = 1 - \frac{b^2}{\mathrm{A}^2m^2 + \mathrm{B}^2}.$$

Hence $\dfrac{X^2}{A^2} + \dfrac{Y^2}{B^2} = 1$ or $A^2Y^2 + B^2X^2 = A^2B^2$.

203. This equation is of the same form as the rectangular central equation, A and B being the intercepts upon the oblique axes. Adding the above values of the squares of conjugate semi-diameters, we have $A^2 + B^2 = \mathrm{A}^2 + \mathrm{B}^2$, as proved in Art. 196; and since m is the tangent of the inclination of B, if we substitute tan θ for m, the value of B^2 will reduce to that of r^2, in Art. 182.

We may now drop the distinction between rectangular and oblique co-ordinates, and regard $\mathrm{A}^2\mathrm{Y}^2 + \mathrm{B}^2\mathrm{X}^2 = \mathrm{A}^2\mathrm{B}^2$ as the equation of an ellipse, referred to any pair of conjugate diameters. From the form of the equation, it is evident, that if any numerical values of X and Y satisfy the equation, four points of the curve may be found having these values, either positive or negative, as co-ordinates. Thus, let a and b represent the numerical values, then the four points (a, b), $(-a, b)$, $(-a, -b)$ and $(a, -b)$ are all on the curve. The lines joining these points consecutively are parallel to the co-ordinate axes, and form a parallelogram inscribed in the ellipse, and the diagonals of the parallelogram are diameters of the ellipse. Two adjacent sides, which together subtend half the curve, are called *supplementary chords* of the ellipse. Therefore, *supplementary chords are parallel to a pair of conjugate diameters.*

In the circle, every pair of supplementary chords is at right angles; but in the ellipse, perpendicular supplementary chords are parallel to the axes, because they form the only pair of rectangular conjugate diameters. When the centre of an ellipse is known, the axes may be constructed geometrically thus: Describe a circle concentric with the ellipse, and cutting the ellipse in four points; the line joining two opposite points will be a diameter, and the lines joining its extremities with one of the other points, will be supplementary chords, both of the circle and ellipse. Hence they are at right angles, and parallel to the axes of the ellipse.

Similar Ellipses.

204. Ellipses having their axes proportional are said to be *similar*, because the ratio of the axes determines the shape of an ellipse. Similar ellipses have the same eccentricity, for the value of e depends upon the ratio of the axes.

By the value of the square of a semi-diameter, Art. 182,

$$r^2 = \frac{B^2}{1 - e^2 \cos^2\theta},$$

we see that *the semi-diameters of similar ellipses making the same angles with their axes, are proportional to the axes.* Therefore, if the major axes of similar ellipses are parallel, all the parallel diameters have the same ratio.

Now the value of mm', the product of the tangents of the inclinations of conjugate diameters to the major axis, is $-\frac{B^2}{A^2}$; it is therefore the same for similar ellipses. Hence, the axes being parallel, *the parallel diameters of similar ellipses have parallel conjugates.*

Again, if two ellipses are such that every pair of conjugate diameters in one is parallel to a pair of conjugate diameters in the other, they are similar; for the axes will then be parallel (since they are the only rectangular conjugate diameters), and the value of mm' will be the same, and hence the ratio of the axes is the same in the two ellipses.

205. Since the equation $A^2Y^2 + B^2X^2 = A^2B^2$ has now been shown to represent an ellipse, even when the axes are oblique, the reasoning of Art. 189 applies to an ellipse referred to conjugate diameters, and a straight line whose direction ratio is m. The value of m', the direction ratio of the conjugate diameter, will be of the same form, and we have the general relation,

$$mm' = -\frac{B^2}{A^2},$$

between *the direction ratios* of any pair of conjugate diameters,* as referred to the pair whose lengths are 2A and 2B.

Now, *let the ratio* A : B *be the same in the equations of two ellipses,* referred to the same or parallel axes; then the values of m', corresponding to the same value of m, will be the same; that is, all parallel diameters in the two ellipses will have parallel conjugates. Therefore by the last Article, *the ellipses will be similar and their axes will be parallel.* Thus, the equations $2x^2 + 3y^2 = 6$, and $2x^2 + 3y^2 = 24$, when reduced to the form $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, are $\frac{x^2}{3} + \frac{y^2}{2} = 1$ and $\frac{x^2}{12} = \frac{y^2}{8} = 1$. The ratio 2 : 3 being the same as 12 : 8, the equations represent similar ellipses, whatever the inclination of the axes.

206. Let $n = \frac{A}{B}$, then the central equation of the ellipse may be written in the form,

$$X^2 + n^2 Y^2 = A^2,$$

which is a formula for the ellipse in terms of its intercept on the axis of X, and an abstract number or ratio, n, which determines its shape. The shape of the ellipse, however, depends not only upon the value of n, but also upon the inclination of the co-ordinate axes; that is, the angle between the conjugate diameters whose ratio is n. Thus, if $n = 1$, the equation reduces to

$$X^2 + Y^2 = A^2,$$

which represents an ellipse referred to equal conjugate diameters,† but not a circle unless the axes are rectangular.

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* If B = A, we have $mm' = -1$. This is the general relation between the direction ratios of lines parallel to the conjugate diameters of an ellipse making equal intercepts on the axes. When the axes are rectangular, the lines are perpendicular.

† *The acute angle between the equal pair is less than that between any other conjugate diameters of an ellipse.* For the tangent of the mutual inclination of any pair referred to the axes of the curve is $\frac{m - m'}{1 + mm'}$. (See note to Art. 47.) Now mm' is constant, and since m and m' are of opposite signs, we may consider m as positive. Therefore the denominator of this fraction is constant, and the numerator is the sum of two quantities whose product is

In the rectangular equation of the ellipse, we considered B as less than A, because we referred the ellipse to its major axis, as the axis of X; but hereafter, since A may denote any semi-diameter taken as the axis of X, we may have $A < B$, and n may have any value greater or less than one.

If $n = 0$, the equation reduces to $X^2 = A^2$, equivalent to $X = A$ and $X = - A$, the equations of two straight lines parallel to the axis of Y.

If $A = 0$, the equation will be satisfied by no point except the origin; unless at the same time $n = 0$, when the equation reduces to $X^2 = 0$, which represents two straight lines coincident with the axis of Y.

Axes Parallel to Conjugate Diameters.

207. If in the central equation of the ellipse, we substitute for the central co-ordinates, X and Y, their values $(x - x')$, $(y - y')$, we shall have

$$A^2 (y - y')^2 + B^2 (x - x')^2 = A^2B^2.$$

This is the equation of an ellipse with centre at P′, and having a pair of conjugate semi-diameters, equal to A and B, parallel respectively to the axes of X and Y. Thus, the equation of the ellipse whose centre is $(2, - 1)$, with conjugate semi-diameters, 2 and 3 units in length, parallel to the axes, is $4(y + 1)^2 + 9(x - 2)^2 = 36$ or $9x^2 + 4y^2 - 36x + 8y + 4 = 0$.

Using the central equation of the last Article, $X^2 + n^2Y^2 = A^2$, we have

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constant. The least value of such a sum occurs when the quantities are equal; hence the tangent of the inclination or the acute angle between two conjugate diameters is least, when $m = - m' = \frac{B}{A}$; that is, for the equal pair. This acute angle is twice the angle whose tangent is $\frac{B}{A}$, and is therefore the smaller in the more eccentric ellipse. The more oblique the axes, when $n = 1$, the more eccentric the ellipse; and the ellipse $X^2 + Y^2 = A^2$ is the *least* eccentric possible, the inclination of the axes being given, for if n is not unity, the obliquity of the equal conjugate diameters is greater than that of the axes.

$$(x-x')^2+n^2(y-y')^2=\mathrm{A}^2,$$

in which n determines the shape of the ellipse, A its size, and x', y' the position of its centre. Expanding the equation,

$$x^2+n^2y^2-2x'x-2n^2y'y+x'^2+n^2y'^2-\mathrm{A}^2=0.$$

Any equation of the general form,

$$\mathrm{A}x^2+\mathrm{C}y^2+\mathrm{D}x+\mathrm{E}y+\mathrm{F}=0,$$

in which A and C have finite values, may be reduced to the form

$$x^2+cy^2+dx+ey+f=0,$$

where c, d, e and f stand for the ratios of the coefficients. Comparing this with the expanded form, we see that it is the equation of an ellipse, if c is put for n^2, d for $-2x'$, e for $-2n^2y'$ and f for $x'^2+n^2y'^2-\mathrm{A}^2$. Since n^2 is essentially positive, c must be positive in order that an equation of this form should represent an ellipse; or, in the general form, *if* A *and* C, *the coefficients of* x^2 *and* y^2, *have the same sign, the equation represents an ellipse.*

208. To determine an ellipse whose equation is given in the form $x^2+cy^2+dx+ey+f=0$; that is, to find the position of its centre, etc., we may compute n^2, x', y' and A^2, by the relations between the constants,

$$n^2=c, \qquad x'=-\tfrac{1}{2}d, \qquad y'=-\tfrac{1}{2}\frac{e}{n^2}=-\frac{e}{2c},$$

and $$\mathrm{A}^2=x'^2+n^2y'^2-f=\tfrac{1}{4}\left(d_2+\frac{e^2}{c}\right)-f.$$

Thus, given the equation $-2x^2-4y^2+20x-12y-9=0$, which is in the general form; dividing through by -2, the coefficient of x^2, gives $x^2+2y^2-10x+6y+4\tfrac{1}{2}=0$, in which $c=2$, $d=-10$, $e=6$ and $f=4\tfrac{1}{2}$. Therefore $n^2=2$, $x'=5$, $y'=-1\tfrac{1}{2}$, and $\mathrm{A}^2=25$. Hence the centre is the point $(5, -1\tfrac{1}{2})$, the semi-diameter parallel to the axis of X is 5; and since $n=\frac{\mathrm{A}}{\mathrm{B}}$ or $\mathrm{B}=\frac{\mathrm{A}}{n}$, the conjugate semi-diameter parallel to the axis of Y is $2\tfrac{1}{2}\sqrt{2}$.

EXAMPLES.—Determine the ellipse $2x^2 + 3y^2 + 8x - 6y + 10 = 0$; $(3 + x)(x - 2) = (5 - y)(2y - 1)$; etc.

209. The equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ contains all the terms belonging to the general equation of the second degree, except that containing xy, the product of the variables. If $A = 0$, or if $C = 0$, it reduces to the equation of a parabola with its diameters parallel to one or other of the co-ordinate axes. (See Arts. 154 and 161.) If both $A = 0$ and $C = 0$, it ceases to be of the second degree, and represents a straight line. But if A and C have finite values *of the same sign*, it represents an ellipse. This restriction is necessary in order that n, the ratio which determines the shape, should be possible.

Since x' and y' are determined from c, d and e, by expressions of the first degree, the centre P′ can always be found; but if the value found for A^2 is negative, A is *imaginary*, and no ellipse can be constructed. In that case, the equation may be reduced to the form $(x - x')^2 + n^2 (y - y')^2 =$ *a negative quantity*, which can be satisfied by *no* values of x and y, because the sum of two squares is an essentially positive quantity. But, since the equation comes under the form of the equation of the ellipse, and is subject to the above restriction on the signs of the coefficients, it is said to represent an *imaginary ellipse.*

210. If the value found for A^2 is zero, the equation may be reduced to the form

$$(x - x')^2 + n^2 (y - y')^2 = 0,$$

an equation satisfied only by the point P′. Such an equation is said to represent an *infinitesimal ellipse.* Since the values of x', y' and n^2 are independent of f, the absolute term, we may suppose this term to vary, without affecting the position of the centre or the shape of the ellipse. Now from the value of A^2, it appears that if f is negative or zero, A is real and the ellipse real; but if f becomes positive and increases, A decreases until $f = \frac{1}{4}\left(d^2 + \frac{e^2}{c}\right)$, when $A = 0$, and the ellipse *vanishes into a single point.* If f increase beyond this value, A becomes imaginary.

Thus we see that the general equation of Art. 207 (like the general equation of the circle) includes certain equations which are satisfied only by a single point, and others which are satisfied by no points. Since it contains five coefficients, whose ratios c, d, e and f are four arbitrary constants, an equation of this form may be found whose locus shall pass through four given points. But the locus will not always be an ellipse; and therefore the further discussion of the equation is deferred until we have shown what it represents in case the coefficients C and A have contrary signs.

Similar and Parallel Ellipses.

211. It must be remembered that the equation

$$x^2 + cy^2 + dx + ey + f = 0$$

does not include the equations of *all* ellipses, but only such as have a pair of conjugate diameters parallel to the co-ordinate axes. The centre may be in any position, and these diameters of any length; thus, when the equation is in this form, the ellipse is determined by *four* quantities, as in Art. 208. It will be shown hereafter, that in general, *five* quantities or five conditions are necessary to determine an ellipse; one condition being fulfilled by assuming the equation in the above form. Suppose it to be required that the diameters parallel to the axes shall have a given ratio, n. This second condition determines c, since $c = n^2$; hence we may write the equation,

$$x^2 + n^2y^2 + dx + ey + f = 0,$$

where d, e and f are arbitrary, for an ellipse fulfilling the two conditions. Two ellipses, in which the value of n is the same, may be called *similar and parallel* ellipses, because, by Art. 205, their axes, are parallel.

212. For an ellipse similar and parallel to a given ellipse, n will be known; and d, e and f may be determined by three equations of condition, so as to make the ellipse pass through three given points, just as a circle was found passing through three given points in Art. 110, and a parabola, in Art. 159. Therefore an ellipse, similar and parallel to a given ellipse, may be found passing through any three points, except when they are in the same straight line, as explained in Art. 159, in the case of the parabola with the direction of its axis determined.

All circles are similar and parallel ellipses, because any perpendicular diameters may be taken as axes, and the ratio of diameters is always one of equality. Parallel parabolas may be regarded as the limiting case of similar and parallel ellipses, in which n becomes zero or infinite.*

213. Every straight line may be said to cut an ellipse in two real, coincident or imaginary points, according to the character of the values of x and y, found by elimination between their equations. Two ellipses evidently may intersect in four points, but *similar and parallel ellipses, can only intersect in two points.* For let n be the ratio determining their shape, when they are referred to axes parallel to co-ordinate diameters of each, then their equations will be of the form $x^2 + n^2y^2 + dx + ey + f = 0$ and $x^2 + n^2y^2 + d'x + e'y + f' = 0$. The equation formed by combining these is

$$x^2 + n^2y^2 + dx + ey + f + k\,(x^2 + n^2y^2 + d'x + e'y + f') = 0.$$

which represents generally an ellipse similar and parallel to the given ellipses, because, dividing throughout by $1 + k$, the terms of the second degree become $x^2 + n^2y^2$. The locus of the equation passes through all the points of intersection of the original ellipses. But when $k = -1$ the equation reduces to

$$(d - d')\,x + (e - e')\,y + f - f' = 0,$$

an equation of the first degree, whose locus, being a straight line, can only intersect either ellipse in two points. Therefore the ellipses have but two points of intersection. In other words, we can eliminate at once both the terms of the second degree from the equations, and then having an equation of the first degree to combine with either, we can find the intersections.

EXAMPLES.—Find the intersections of $2x^2 + 3y^2 + 8x - 6y - 10 = 0$ with $2x^2 + 3y^2 + 6x - 4 = 0$; of $x^2 + y^2 + 2x - 2y +$

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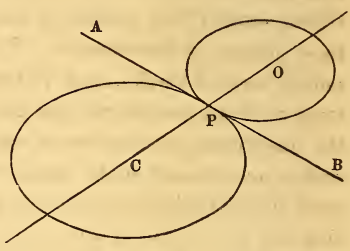
* The equation of the ellipse referred to a diameter and tangent is $y^2 + \frac{B^2}{A^2}x^2 = 2\frac{B^2}{A}x$. Put $\frac{B^2}{A} = p$, then $\frac{B^2}{A^2} = \frac{p}{A}$, and $y^2 + \frac{p}{A}x^2 = 2px$. Let p remain constant, and A and B increase without limit, then $\frac{B^2}{A^2} = \frac{p}{A}$ decreases to zero, and we have the equation of the parabola. Thus the ellipse becomes a parabola when A and B are both infinite, and their ratio $\frac{B}{A}$ is zero, or $\frac{A}{B}$ is infinite.

$1=0$ with $2x^2+2y^2-4x-4y+2=0$; of $3x^2+y^2-6x=8$ with $2y^2=12x-6x^2$.

214. If the ellipses touch one another, the straight line will be a common tangent, as AB in the figure, the two points of intersection coinciding at P. Join P to the centres, O and C, then CP and OP are semi-diameters of the two ellipses, conjugate to diameters parallel to AB. But since parallel diameters have parallel conjugates, Art. 204, CP and OP must form a single straight line. Hence *if similar and parallel ellipses touch, the point of contact is in a straight line with their centres.*



If the ellipses cut each other, the straight line AB will cut the ellipses in two points P,P, the extremities of a common chord, and if we join the centres of the ellipses with the middle point of the chord, we shall construct diameters conjugate to the diameters parallel to AB, which will form a single straight line as before. Hence *a common chord to similar and parallel ellipses is bisected by the line joining their centres.*

When the ellipses do not meet, the line AB whose equation is

$$(d-d')\,x+(e-e')\,y+f-f'=0,$$

still has a definite position, and is *parallel to the diameters conjugate to* OC. For the centres were found, in Art. 208, to be the points $\left(-\tfrac{1}{2}d, -\tfrac{1}{2}\dfrac{e}{n^2}\right)$ and $\left(-\tfrac{1}{2}d', -\tfrac{1}{2}\dfrac{e'}{n^2}\right)$. The direction ratio of the line passing through these points, or the difference of their ordinates divided by the difference of their abscissas (Art. 65), is $\dfrac{e-e'}{n^2(d-d')}$. Multiplying this by $-\dfrac{d-d'}{e-e'}$, the direction ratio of AB, we have $-\dfrac{1}{n^2}$ or $-\dfrac{B^2}{A^2}$, since n^2 was put for $\dfrac{A^2}{B^2}$ in each ellipse. The product of the direction ratios of AB and OC, therefore, equals the product of the direction ratios of the diameter OC and its conjugate in either ellipse; hence the latter is parallel to AB.

215. We have seen that the equation formed by combining the equations of similar and parallel ellipses represents generally an ellipse similar and parallel to the given ones, and passing through their points of intersection. The line AB is analogous to the radical axis of circles, and is common to the whole system; and since we have just shown that this line determines the direction of the line joining the centres of *any* two of them, therefore *all their centres are situated on the* same straight line. The arbitrary constant k may be determined by an equation of condition, so as to find that one of the system which passes through a given point.

If $d = d'$ and $e = e'$, that is, if the given equations differ only in the absolute term, the ellipses are concentric. The line AB will then be at infinity, for its equation will take the impossible form (see Art. 43); and since the centres coincide, OC will be indeterminate in direction. The combined equation in this case will represent a series of similar ellipses whose axes coincide.

EXAMPLES.—Find a similar ellipse passing through the intersection of $2x^2 + 3y^2 + 8x - 6y - 10 = 0$ and $2x^2 + 3y^2 + 6x - 4 = 0$, and also through $(3, 1)$.

Find ellipses similar and concentric with one of the above; 1st, passing through $(6, -1)$; 2d, through the origin.

What is represented by the equation

$$(x - x')^2 + n^2 (y - y')^2 + k [y - y' - m (x - x')] = 0?$$

Ans. An ellipse whose shape is determined by n^2, and which is tangent at P′ to the straight $y - y' = m (x - x')$; for the first of the equations combined is satisfied only by the single point P′.

By this formula, supposing the axes rectangular, give the equation of a circle touching $2x = 3y - 1$, at $(1, 1)$, and passing through $(2, 3)$.

Give the equation of the locus of the centres of the ellipses represented by the above equation.

Ans. $y - y' = -\dfrac{1}{mn^2}(x - x')$; for it passes through P′ and $mm' = -\dfrac{1}{n^2}$.

Give the general equation of the ellipse similar to $4y^2 + 2x^2 = 10$, and tangent to $4y = x$, at $(4, 1)$; also the locus of its centre.

Tangent at a Given Point.

216. Since $A^2Y^2 + B^2X^2 = A^2B^2$ represents an ellipse, whatever the inclination of the axes, the expressions found in Art. 189, for the intersections of $Y = mX + b$ with the ellipse, are of general application. The condition of tangency is therefore the same, and the equation of the tangent and co-ordinates of its point of contact are of the same form. That is,

$$Y = mX \pm \sqrt{A^2m^2 + B^2}$$

is tangent to the ellipse at P_1, whose co-ordinates are

$$X_1 = \frac{-A^2m}{\pm\sqrt{A^2m^2 + B^2}} \quad \text{and} \quad Y_1 = \frac{B^2}{\pm\sqrt{A^2m^2 + B^2}}.$$

The double sign shows that there are two tangents having the direction ratio m, and the same sign must be given to the radical in the values of X_1 and Y_1 as in the equation of the tangent. Whichever sign is taken, the constant or value of b in the equation, equals $\frac{B^2}{Y_1}$, and $m = -\frac{B^2X_1}{A^2Y_1}$. Making these substitutions, and clearing of fractions, we have

$$A^2YY_1 + B^2XX_1 = A^2B^2,$$

which is the equation of the tangent expressed in terms of the co-ordinates of its point of contact.

217. Since this equation contains two constants, X_1 and Y_1, in place of m, it may represent *any* straight line if they are considered arbitrary or independent; and only represents a tangent when they are connected by the relation

$$A^2Y_1^2 + B^2X_1^2 = A^2B^2;$$

that is, when P_1 is a point of the ellipse. This relation is therefore the condition of tangency.

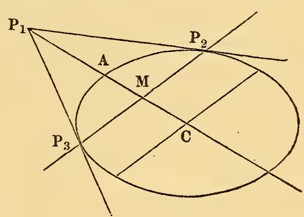
In general, the line is called the polar of the point P_1, with respect to the ellipse, as in the case of the corresponding formulæ for the circle and parabola. The equation

$$A^2Y_2Y_1 + B^2X_2X_1 = A^2B^2$$

then expresses either that P_2 is on the polar of P_1 or that P_1 is on the polar of P_2, hence points connected by this relation are said to be *reciprocally polar*. A point on its own polar, or *self-polar* point is a point of the curve, and its polar is a tangent, as in Arts. 124 and 174.

From these properties it follows, as in the Articles referred to, that the points of contact for tangents passing through a given point are the intersections of the curve with the polar of the point. For the given point is on the polar of the required point (that is, the required tangent); hence the required points are on the polar of the given point.

218. The polar of every point on a given diameter of the ellipse is parallel to the conjugate diameter. For the direction ratio of the polar is $-\frac{B^2X_1}{A^2Y_1}$ which is constant as long as the ratio of X_1 and Y_1 is unchanged; that is, while P_1 is on the diameter CP_1. But the polar of A, the vertex of this diameter, is the tangent at that point, which has been shown to be parallel to the conjugate diameter. This is evidently true, to whatever conjugate diameters the lines be referred, for the equation of CP_1 passing through a given point, and the origin is $Y = \frac{Y_1}{X_1}X$, and the product of its direction ratio and that of the polar is $-\frac{B^2}{A^2}$. Accordingly, if CP_1 is made the axis of X, so that $Y_1 = 0$, the equation of the polar is $XX_1 = A^2$ or $X = \frac{A^2}{X_1}$, showing that it is parallel to the axis of Y. In this equation, A represents the semi-diameter CA, which was made the axis of X, and X_1 represents the distance CP_1. Hence CM is a third proportional to CP_1 and CA. We may construct the polar of a given point by laying off this distance and drawing a parallel to the conjugate diameter.



The chord P_2P_3 is a double ordinate to the diameter CA, and since by the last Article the tangents at P_2 and P_3 pass through P_1, we see that the tangents at the extremities of a double ordinate meet on the diameter produced, and make an intercept which is a third proportional to the abscissa and semi-diameter. Compare Art. 123, remembering that all the conjugate diameters of the circle are at right angles.

EXAMPLES.—Prove the above property of tangents by means of the equation of the tangents referred to CA as axis of X.

Show that the polar of no point can pass through the centre, and that the centre has no polar.

Prove that the polar of the focus is the directrix on the same side of the centre; and show that therefore the tangents at the extremity of a focal chord meet in the directrix.

The polar of a point without the curve is a secant; find the condition that P_1 should be without the curve, by applying condition of secancy, Art. 189, to its polar. (Reduce the polar to the form $Y = mX + b$, and assume the quantity under the radical sign in the values of the co-ordinates of intersection to be positive.)

219. Substituting for the central co-ordinates, X, X_1, etc., their values $(x - x')$, $(x_1 - x')$, etc., we have

$$A^2 (y - y') (y_1 - y') + B^2 (x - x') (x_1 - x') = A^2B^2,$$

for the polar of P_1, with respect to the ellipse whose centre is P'; or dividing by B^2, and putting n for the ratio $\frac{A}{B}$,

$$(x - x') (x_1 - x') + n^2 (y - y') (y_1 - y') = A^2.$$

Expanding, the equation becomes

$$xx_1 + n^2yy_1 - x' (x + x_1) - n^2y' (y + y_1) + x'^2 + n^2y'^2 - A^2 = 0,$$

and introducing the constants, c, d, e and f with the same values as in Art. 207, we have

$$xx_1 + cyy_1 + \tfrac{1}{2}d (x + x_1) + \tfrac{1}{2}e (y + y_1) + f = 0,$$

for the polar with respect to the ellipse,

$$x^2 + cy^2 + dx + ey + f = 0.$$

We therefore produce the formula for the tangent or polar with respect to an ellipse whose equation is given, in the same way as in the previous cases of the circle and parabola; namely, by substituting xx_1 for x^2 in the equation of the curve, yy_1 for y^2, $\frac{1}{2}(x + x_1)$ for x and $\frac{1}{2}(y + y_1)$ for y. Thus, given the ellipse $4x^2 + 9y^2 - 8x + 18y - 167 = 0$, the formula for a polar is $4xx_1 + 9yy_1 - 4x - 4x_1 + 9y + 9y_1 - 167 = 0$, or

$$(4x_1 - 4)\,x + (9y_1 + 9)\,y - 4x_1 + 9y_1 - 167 = 0.$$

EXAMPLES.—Find tangents to the above ellipse passing through the point $(-4, 2\frac{1}{3})$; through $(16, -1)$. (The required tangents are the polars of the intersections of the curve with the polars of the given points.)

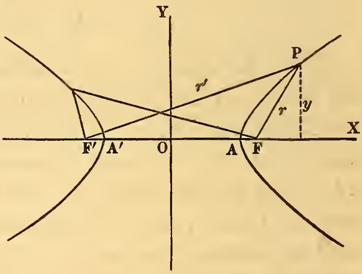
Show, by the general equation, that the centre generally has no polar, and that the polar of no point can pass through the centre; but that when the ellipse is infinitesimal, every polar passes through the centre. (See Art. 208 for co-ordinates of the centre.)

CHAPTER VII.

THE HYPERBOLA.

220. If a point move in such a manner that the difference of its distances from two fixed points is constant, it will describe a curve called an *hyperbola*. The two fixed points are called the *foci*.

To find the equation of the hyperbola, take for the axis of X, the straight line passing through the foci, F and F′; and for the axis of Y, a perpendicular line bisecting F′F. Let c denote the distance from either focus to the origin; and 2A, the constant difference of the lines PF′ and PF, or r' and r, the *focal distances* of the point P.



Then, drawing the ordinate of P, we have from right-angled triangles, the geometrical relations,

$$r'^2 = y^2 + (x + c)^2 \qquad \text{and} \qquad r^2 = y^2 + (x - c)^2;$$

and from the definition of the curve,

$$r' - r = 2\text{A}.$$

We have now to combine these three equations so as to derive an equation between x, y and the constants. From the first two we have $r'^2 - r^2 = 4cx$; dividing by $r' - r = 2\text{A}$, $r' + r = \dfrac{2cx}{\text{A}}$. Hence

$$r' = \frac{cx}{\text{A}} + \text{A} \qquad \text{and} \qquad r = \frac{cx}{\text{A}} - \text{A}.$$

Finally, substituting the value of r' in the first equation, or that of r in the second, we have

$$\frac{c^2x^2}{A^2} + A^2 = y^2 + x^2 + c^2 \quad \text{or} \quad \frac{c^2 - A^2}{A^2}\,x^2 - y^2 = c^2 - A^2.$$

221. This is the equation of the hyperbola in terms of the constants A and c. Like the corresponding equation of the ellipse, it will take a more convenient form when we introduce another constant. Since in this case c is necessarily greater than A, $c^2 - A^2$ is a positive quantity, and may be denoted by B^2. Making the substitution, and dividing by B^2, we find

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1.$$

Now if we make $y = 0$, we have $x_0{}^2 = A^2$ or $x_0 = \pm A$; that is, the curve cuts the axis of X in two points, A and A′, on the right and left of the origin. The focal distances of A are $r' = A + c$ and $r = A - c$, of which r' *exceeds* r by the required difference 2A. But for the point A′, r' is less than r; it would therefore seem as if that point did not satisfy the condition $r' - r = 2A$, used in deducing the equation. We shall find, however, by examining the analytical expressions for r and r', that the condition is satisfied by A′. Putting e in place of the ratio $\frac{c}{A}$, which is called the *eccentricity* of the hyperbola, the expressions of the last Article become

$$r' = ex + A \qquad \text{and} \qquad r = ex - A.$$

Since $c > A$ for the hyperbola, $e > 1$. Now for the point A′, $x = -A$, therefore the values, both of r' and of r, are *negative*, and r', which has the least *numerical* value, still exceeds r *algebraically* by the required difference 2A. For every negative value of x numerically greater than A, we have, in like manner, *negative values of the focal distances*, of which that of r' exceeds that of r algebraically by the constant difference 2A. Hence *the hyperbola consists of two branches*, one on the right and one on the left of the axis of Y.

222. The part A′A, of the line joining the foci, is called the

transverse axis of the curve, A and A′ are its *vertices*. The middle point, O, is called the *centre*. In investigating the form and properties of the curve we shall use X and Y, as in Chapters IV. and VI., to denote central co-ordinates. Then (clearing of fractions, etc.),

$$A^2Y^2 - B^2X^2 = - A^2B^2$$

is the *central equation* of the hyperbola, and

$$A^2 (y - y')^2 - B^2 (x - x')^2 = - A^2B^2$$

is the rectangular equation of the hyperbola with centre at P′ and transverse axis parallel to the axis of X.

The equation of the hyperbola in terms of A and c, Art. 220, is of the same form as that of the ellipse, Art. 177, the difference being only in the comparative values of A and c. But, since B^2 is in this case put for $c^2 - A^2$, whereas in the case of the ellipse it was put for $A^2 - c^2$, the equations containing B differ from the corresponding ones of the ellipse in the sign prefixed to B^2. The value of c is determined by the equation

$$c^2 = A^2 + B^2.$$

Whatever the values of A and B in this equation, c is a real quantity; therefore the above central equation *always* represents an hyperbola having its foci on the axis of X.

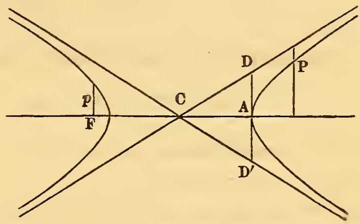
Form of the Hyperbola.

223. Making Y an explicit function of X, the central equation takes the form

$$Y = \pm \frac{B}{A} \sqrt{X^2 - A^2}.$$

From this we see that there is no point of the curve having an abscissa *numerically* less than A; for such a value of X would make Y imaginary. But for all values of X, positive or negative, numerically greater than A, equal positive and negative values of Y may be found, and points of the curve constructed. Considering now, only positive values of X and Y, Y is an *increasing* function of X. That is, when $X = A$, $Y = 0$; and as X increases, Y increases. Now, since $X^2 - A^2$ is less than X^2. the radical is less than X, and $Y < \frac{B}{A} X$.

Suppose now the values of A and B to be given; construct the point (A, B), by laying off from C a distance equal to A on the axis of X, and erecting a perpendicular equal to B. Join the point D, so found, with the centre, then the equation of CD (passing through this point and the origin) is $Y = \frac{B}{A} X$. As



the value of X increases, the difference between X and the radical $\sqrt{X^2 - A^2}$ continually decreases; for the difference of the squares of these quantities (which is the product of their sum and difference) is constant, and as X increases their sum increases. Hence, the ordinate to the curve is always less than the ordinate to the line CD, corresponding to the same abscissa, but continually approaches to it as the abscissa is increased; or the distance between the straight line and curve decreases without limit, when the abscissa increases without limit.

The lower portion of the same branch of the curve approaches, in like manner, to the line CD′, whose equation is $Y = -\frac{B}{A} X$; and the other branch of the hyperbola is similarly situated within the opposite angle formed by the lines CD and CD′. The form of the curve is, therefore, that represented in the figure.

224. The straight lines CD and CD′, which continually approach the curve, but can never meet it, are called the *asymptotes*. The asymptotes make equal angles with the transverse axis. The ratio B : A determines this angle, and may therefore be considered as determining the *shape* of the hyperbola. The angle between the asymptotes, which is double this angle, also determines the shape of the curve. If $B < A$, as in the figure, the angle DCA, or inclination of an asymptote is less than $45°$; and DCD′, the angle between the asymptotes, is an acute angle. If $B = A$, DCA will be

45°, and DCD′ *a right angle.* In this case the hyperbola is called *equilateral* or *rectangular.* The hyperbola of the figure may be considered *acute*, but if B were greater than A, it would be *obtuse.*

Since the perpendicular sides of the triangle CAD were constructed respectively equal to A and B, and $c^2 = A^2 + B^2$, Art. 222, the hypothenuse $CD = CF$ the distance from the centre to the focus. This triangle may therefore be constructed when the values of A and c are given, as well as when A and B are given. From the trigonometric definitions,

$$\tan DCA = \frac{B}{A}, \qquad \sec DCA = \frac{CD}{CA} = \frac{c}{A} = e,$$

therefore the eccentricity also determines the shape of the curve.

225. The ordinate corresponding to either focus is found by putting $X = \pm c$ or $X^2 = c^2$, which gives $Y = \pm \frac{B^2}{A}$, since $c^2 - A^2 = B^2$. This ordinate is denoted by p, as in the cases of the parabola and ellipse, and the double ordinate passing through the focus or $2p$ is called the *parameter.* The following are the most useful relations between the constants A, B, c, e and p found by combining $e = \frac{c}{A}$, $B^2 = c^2 - A^2$ and $p = \frac{B^2}{A}$:

$$c = A\,e, \qquad B^2 = A^2\,(e^2 - 1), \qquad p = A\,(e^2 - 1).$$

After the shape of the hyperbola, or angle between the asymptotes, is determined by the value of e or by the ratio B : A, its *size* may be determined by A, c or p. Hyperbolas having the same asymptotes being regarded as of the same shape, those which approach nearest to the asymptotes will be the less in size.

By the above relations, the values of the remaining constants may be computed when any two of them are given; and the inclination of the asymptotes may be computed by the trigonometric tables.

Examples.—Find the parameter and the eccentricity when $A = 2$ and $B = 4$.

Find the value of A when $p = 3$ and $e = 2$, and show that the angle between the asymptotes is 120°.

Express B^2 in terms of p and e.

What is the eccentricity of the equilateral hyperbola?

Polar Equations.

226. By the formulæ of transformation, the polar equation of the hyperbola, when the pole is the centre, is

$$r^2 = \frac{-A^2B^2}{A^2 \sin^2 \theta - B^2 \cos^2 \theta}.$$

It is evident that every value of θ which makes this value of r^2 positive will give equal positive and negative values of r. Hence a straight line drawn through the centre meeting one branch of the hyperbola will, if produced, meet the other branch at an equal distance from the centre. Such a line is called a diameter, therefore *a diameter is bisected at the centre.* Putting for B^2, in the denominator, its value $c^2 - A^2$, we have

$$r^2 = \frac{-A^2B^2}{A^2 - c^2 \cos^2 \theta} = \frac{-B^2}{1 - e^2 \cos^2 \theta}.$$

Since the numerator of the last expression is negative, the value of r^2 will be positive, and the value of r real, only when the denominator is also negative. When $\theta = 0°$ this is the case, because $e > 1$ (since $c > A$ for the hyperbola); the result being $r^2 = A^2$, which is its least possible value. As θ increases, the denominator decreases and r^2 increases, until θ reaches such a value that the denominater becomes zero, which takes place when $\cos \theta = \frac{1}{e}$ or $\sec \theta = e$. This value of θ is the inclination of the asymptote, as shown in Art. 224, and it makes r^2 *infinite.* A greater value of θ makes r^2 negative and r imaginary, until we reach a value in the second quadrant for which $\sec \theta = -e$, which again makes r^2 infinite. This value is supplementary to the former, and is therefore the inclination of the other asymptote. Values of θ between this and 180° make r^2 positive and r real; therefore every line passing through the centre, and between either asymptote and the axis, cuts both branches of the curve and is a diameter. The asymptotes are thus the *limits of the diameter*, and all diameters pass through the angle DCD′, or *interior* angle of the asymptotes.

227. The denominator of the above value of r^2 is of the same form as that which occurs in the corresponding equation for the ellipse, Art. 182, and the equations differ only in the sign prefixed to B^2. When either of these curves is referred to a focus or vertex, its equation will be most conveniently expressed in terms of p and e, and it will then be found that the same equation will represent either curve, according as we suppose e greater or less than unity. Also, if we make $e = 1$, these equations will reduce to the corresponding forms for the parabola.

These three curves are included under the general name of the *conic sections.** The size of the conic section may be regarded as determined by p, and its shape by e. The parabola, in which $e = 1$, is of definite shape, and is intermediate between the ellipse in which $e < 1$ and the hyperbola in which $e > 1$. The circle, in which $e = 0$, is a particular case of the ellipse; in its equation R takes the place of p.

228. To show that these curves may be expressed by a general rectangular equation, containing the constants p and e, we refer the hyperbola to the *right-hand focus* F. The centre is, therefore, on the left of the origin, at the point $(-c, 0)$.† Substituting its coordinates for x' and y' in $A^2(y-y')^2 - B^2(x-x')^2 = -A^2B^2$, and expanding, we have

$$A^2y^2 - B^2x^2 - 2B^2cx = -A^2B^2 + B^2c^2 = B^4.$$

If now we divide each member by A^2, and make use of the relations between the constants, Art. 225, we have

$$y^2 + (1-e^2)x^2 - 2pex = p^2,$$

or

$$x^2 + y^2 = (p + ex)^2.$$

The equation of the ellipse in Art. 185 may be reduced to the

* This term is derived from the fact that these are the only curves which can be produced by the intersection of a right cone by a plane.

† The right focus of the hyperbola must be regarded as corresponding to the left focus of the ellipse, so that the nearest vertex will be in each case on the left. Then, as the centre of the hyperbola is on the left, both of the focus and vertex, c and A change signs (passing through infinity) as we pass from the ellipse to the hyperbola. Replacing A, c and B^2, by $-A$, $-c$ and $-B^2$, while p and e continue positive, all the equations of the ellipse become equations of the hyperbola.

same form, by employing the relations between the constants in Art. 183. When $e = 1$, it reduces to $y^2 = 2px + p^2$, found for the parabola in Art. 144; and when $e = 0$, it reduces to $x^2 + y^2 = p^2$ the equation of a circle whose radius is p. Therefore it may represent any conic section.

229. Since $x^2 + y^2 = r^2$, the above equation transformed to polar co-ordinates is $r^2 = (p + er\cos\theta)^2$ or $r = \pm(p + er\cos\theta)$, giving

$$r = \frac{p}{1 - e\cos\theta} \quad \text{or} \quad r = \frac{-p}{1 + e\cos\theta},$$

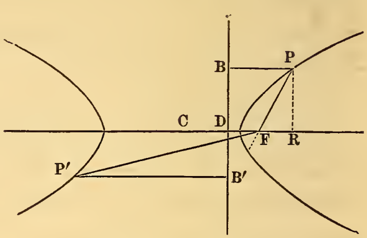
the same two values of r which were found for the ellipse. But, since $e > 1$ for the hyperbola, the first value is no longer always positive, nor the second always negative, as in case of the ellipse. Thus, when $\theta = 0°$ the first value of r is negative, and by the relations between the constants may be proved equal to $-(A + c)$: hence this value of θ gives the vertex of the left branch. As θ increases, the negative value of r increases, describing the lower part of the left branch of the curve, until $\sec\theta = e$, when it becomes infinite. Therefore when r has an inclination equal to that of the asymptote, its value passes through infinity, and becomes positive. As the inclination is increased, the right branch of the curve is described, r remaining positive until we reach the value in the fourth quadrant, for which again $\sec\theta = e$, and r is infinite. Between this value and $\theta = 360°$, r is negative and describes the upper part of the left branch. Thus, during an entire revolution, both branches are described; the right by positive values of r, the left by negative values.

The second value of r may, in like manner, be shown to describe both branches during an entire revolution; but it describes the right branch by negative values, and the left by positive.

230. The first value of r, which is *positive for the branch within which the focus is situated*, might have been derived from the value of r, in Art. 221. Now x in $r = ex - A$ denotes the abscissa CR measured from the centre, and CR is $x + c$, when x represents FR, the abscissa measured from the focus. Therefore $r = ex + ec - A$, or since $c = Ae$ and $A(e^2 - 1) = p$ (Art. 225),

$$r = p + ex.$$

This is the same relation as that shown in Art. 184 to exist between the radius vector and abscissa of the ellipse as referred to its focus; it is equivalent to the first value of r, and applies to all conic sections.*



In the hyperbola, the value of r thus found is positive for the right branch, and negative for the left, as shown in Art. 221.

231. If the perpendicular DB be drawn, at the distance $\mathrm{FD} = \frac{p}{e}$ to the left of the focus, it may be proved (as for the ellipse in Art. 184) that *the distances of a point of the curve, from a fixed point or focus* F, *and a fixed line or directrix* DB, *are in the constant ratio* $e:1$. Thus the three conic sections may be defined by a common property, from which the general equation of the conic section, Art. 228, may easily be derived. For, the origin being at F, $\mathrm{PF}^2 = x^2 + y^2$ and $\mathrm{PB} = \mathrm{DR} = \frac{p}{e} + x$; but by the above property $\mathrm{PF} = e\mathrm{PB} = p + ex$; hence $x^2 + y^2 = (p + ex)^2$.

For the hyperbola, the point is nearer to the directrix than to the focus, and therefore may be found on the left of the directrix, which is impossible for the parabola or ellipse.

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* Since the value of r is not affected by changing the direction of the axes without change of origin, and since this transformation does not alter the absolute term, every equation of the form

$$r + mx + ny = p;$$

that is, every linar equation between r, x and y, represents a conic section, of which the origin is a focus and the absolute term is the semi-parameter. Supposing the axes rectangular, the formulæ of Art. 88 give for the equation, when the axes are turned through the angle a, $r + (m \cos a + n \sin a)\, x + (n \cos a - m \sin a)\, y = p$. The term containing y disappears when $\tan a = \frac{n}{m}$, and the equation reduces to $r + ex = p$, in which $e = \pm \sqrt{m^2 + n^2}$ according to which of the two values (differing by 180°) we take for a. When e is made positive, positive x is measured toward the nearest vertex.

232. The length of the focal chord, in any conic section, is the algebraic difference between the two values of r corresponding to the same value of θ. Thus, subtracting the second value in Art. 229, from the first, we have

$$\frac{p}{1 - e\cos\theta} - \frac{-p}{1 + e\cos\theta} = \frac{2p}{1 - e^2\cos^2\theta}.$$

In the ellipse, the expression thus obtained is always positive, because the first value of r was always positive, and the second always negative, and therefore changing the sign of the second we have two positive quantities to add. But in the hyperbola, the first value is positive for the right branch, and negative for the left; and so is the second value after its sign is changed. (See Art. 229.) Therefore, for a chord which cuts a single branch, as PF produced, the expression is found to be positive; but for a chord which cuts both branches, as the exterior part of P′F, it is negative. Chords parallel to either asymptote are infinite, and a chord parallel to a diameter (which, according to Art 226, has a less inclination to the axis, so that $\sec^2\theta < e^2$) must be regarded as negative.

EXAMPLES.—In the conic section for which $p = 3$ and $e = 1\frac{1}{2}$, find the focal radius vector for $\theta = 60°$; find also the focal chords for $\theta = 30°$, $\theta = 45°$, and $\theta = 60°$.

If a line drawn through the focus, parallel to an asymptote, meet the hyperbola in P, show that PF equals $\frac{1}{4}$ of the parameter, and that the foot of the ordinate of P is midway between the focus and directrix.

Show that the distance of the directrix from the centre is $\frac{A}{e}$.

Prove that the distance of a point of the hyperbola from the focus is equal to its distance from the directrix measured on a line parallel to an asymptote.

Find the distance of the centre of $x^2 + y^2 = (p + ex)^2$ from the focus; and discuss its position, supposing p to be fixed, and e to vary from a large positive to a large negative value. (The centre is midway between the points in which the curve cuts the axis of X.)

Find a rectangular and a polar equation, in terms of p and e, for the hyperbola referred to the right-hand vertex; and show that they apply to all conic sections.

CONJUGATE HYPERBOLAS.

233. The equation of an hyperbola, whose foci are situated on the axis of Y, might be found by a process similar to that of Art. 220, in which x would take the place of y, and y that of x. Hence we infer that the result would be of the same form as if we made this interchange at once in the equation $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$, the result of the previous process. If at the same time we interchange A and B, we shall have

$$\frac{y^2}{B^2} - \frac{x^2}{A^2} = 1,$$

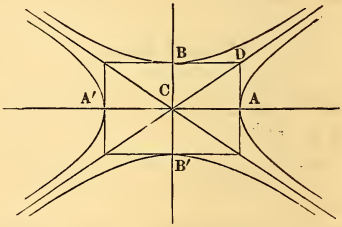
which represents an hyperbola of which B *is the transverse semi-axis measured on the axis of* Y. Accordingly, if $x = 0$ we have $y_0 = \pm B$, but $y = 0$ gives imaginary intercepts on the axis of X.

This hyperbola is called the *conjugate* of $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$. Using X and Y as before for central co-ordinates, we have for its central equation, after clearing of fractions,

$$A^2Y^2 - B^2X^2 = A^2B^2,$$

which differs from the other equation only in the sign of the second member. B, which is the transverse semi-axis of the conjugate hyperbola, is called the *conjugate semi-axis* of the original hyperbola.

234. To construct the asymptotes of the conjugate hyperbola, we should have to construct the same point (A, B), as in Art. 223, laying off from C a distance equal to B on the axis of Y, and erecting a perpendicular equal to A. Hence the two hyperbolas have common asymptotes, which are the diagonals of a rectangle whose centre is C, and whose sides are equal to 2A and 2B. The lines AA′ and BB′ are called the *axes* of both hyperbolas.



The distance CD, in the figure, is equal to the distance of the foci of both hyperbolas from the centre, because $c^2 = A^2 + B^2$, and is therefore the same for each. But the eccentricity of the conjugate hyperbola is the secant of DCB, or the cosecant of DCA whose secant is e. Therefore the two hyperbolas have different shapes. In fact, being situated in the supplemental angles of the asymptotes, one of them is acute and the other obtuse, except when the asymptotes are at right angles, and then both are rectangular or equilateral.

235. The equations of the two asymptotes are $Y = \frac{B}{A}X$ and $Y = -\frac{B}{A}X$, or $AY - BX = 0$ and $AY + BX = 0$. The compound equation, representing at once all the points on both these lines, is by Art. 81,

$$A^2Y^2 - B^2X^2 = 0.$$

This is the equation of the asymptotes. Solving it for Y gives $Y = \pm\frac{B}{A}X$ equivalent to the two equations representing the asymptotes separately. The equation of the asymptotes differs from the equation of either hyperbola only in having zero for its second member.

236. In order to express the two co-ordinates of a point on the ellipse in terms of a single variable, we made use in Art. 187 of the trigonometric formula $\sin^2 + \cos^2 = 1$, because in the equation of the ellipse the sum of two squares is unity. For the hyperbola we make a similar use of the formula $\sec^2 - \tan^2 = 1$. Thus, in the equation $\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1$, $\frac{X}{A}$ and $\frac{Y}{B}$ are the secant and tangent of the same angle. Denoting this angle by ψ, we have

$$X = A \sec \psi \qquad \text{and} \qquad Y = B \tan \psi,$$

which satisfy the equation of the hyperbola whatever be the value of ψ.

The application of this auxiliary angle is similar to that of the eccentric angle in case of the ellipse, although it cannot be constructed geometrically for a given point with the same facility. It may, however, be shown by the algebraic signs of the functions, that values of ψ in the first quadrant correspond to points on the upper part of the right branch; values in the second quadrant correspond to the lower part of the left branch; values in the third, to the upper part of the left; and values in the fourth, to the lower part of the right branch. $0°$ and $180°$ correspond to the vertices A and A′; and $90°$ and $270°$, for which the secant and tangent are both infinite, may be regarded as corresponding to points at infinity in the directions of the asymptotes $AY = BX$ and $AY = -BX$.

For a point on the *conjugate hyperbola*, we may in like manner assume $X = A \tan \psi$ and $Y = B \sec \psi$, for these values always satisfy the equation $\frac{Y^2}{B^2} - \frac{X^2}{A^2} = 1$.

EXAMPLES.—What are the co-ordinates of the point for which $\psi = 45°$?

Ans. $X = A\sqrt{2}$, and $Y = B$. The abscissa being the diagonal of a square constructed on the semi-transverse axis, this point may be constructed when the curve and its axis are given, in order to find a line equal to B.

Prove that the value of ψ for the extremity of the parameter is equal to the inclination of the asymptote.*

SECANT AND TANGENT LINES.

237. To find general expressions for the intersections of a straight line with an hyperbola, we substitute $Y = mX + b$ in the central equation, $A^2Y^2 - B^2X^2 = -A^2B^2$.

Thus, $$(A^2m^2 - B^2)X^2 + 2A^2mbX = -A^2B^2 - A^2b^2$$

is the equation giving the abscissas of the points of intersection. This is generally an equation of the second degree; but when $A^2m^2 - B^2 = 0$, it becomes an equation of the first degree, and gives a single value of X. Therefore, when $A^2m^2 = B^2$, or $m = \pm \frac{B}{A}$; that is, *when the line is parallel to either asymptote, it*

* This angle, whose secant is e, is used as an *auxiliary constant* in astronomical computations. Thus if $e = \sec \psi$, $B = A \tan \psi$, and $p = A(e^2 - 1) = A \tan^2 \psi$. Similarly in the ellipse, if $e = \cos \phi$, $B = A \sin \phi$, and $p = A \sin^2 \phi$. See example under Art. 187.

cuts the hyperbola in a single point. If $m = \frac{B}{A}$, we find for the co-ordinates of this point, $X = -\frac{1}{2} A \left\{ \frac{B}{b} + \frac{b}{B} \right\}$ and $Y = \frac{1}{2} \left\{ b - \frac{B^2}{b} \right\}$, which become infinite when $b = 0$. Therefore the asymptote itself does not meet the curve.

238. Dividing the above equation by the coefficient of X^2, and completing the square,

$$\left(X + \frac{A^2mb}{A^2m^2 - B^2}\right)^2 = \frac{-A^2B^2(A^2m^2 - B^2) + A^2B^2b^2}{(A^2m^2 - B^2)^2}.$$

Hence we have for the general values of the co-ordinates of intersection,

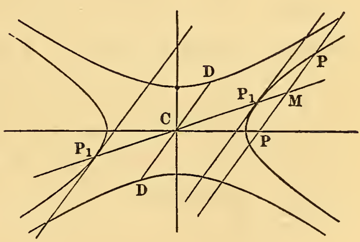
$$X = \frac{-A^2mb \pm AB\sqrt{B^2 - A^2m^2 + b^2}}{A^2m^2 - B^2}$$

and
$$Y = \frac{-B^2b \pm mAB\sqrt{B^2 - A^2m^2 + b^2}}{A^2m^2 - B^2},$$

since $Y = mX + b$.

The rational parts of these values,

$$-\frac{A^2mb}{A^2m^2 - B^2} \quad \text{and} \quad -\frac{B^2b}{A^2m^2 - B^2},$$



are the co-ordinates of M, the middle point of the chord PP. For a given value of m, the ratio of these co-ordinates is constant; for they satisfy the relation $\frac{Y}{X} = \frac{B^2}{mA^2}$, or putting m' for $\frac{B^2}{mA^2}$, $Y = m'X$. But this is the equation of a straight line passing through the centre, or of the diameter MC in the figure. Hence, *a system of parallel chords of the hyperbola is bisected by a diameter.*

239. The condition of tangency is $b^2 = A^2m^2 - B^2$, because this value of b^2 makes the radical equal zero. When this condition is

fulfilled, the points P, P and M will coincide at P_1 the extremity of the diameter MC. Substituting the value $b = \pm\sqrt{A^2m^2 - B^2}$, therefore, in the equation of the straight line, and in the co-ordinates of M, we have

$$Y = mX \pm \sqrt{A^2m^2 - B^2}$$

for the equation of a tangent line, and

$$X_1 = \frac{-A^2m}{\pm\sqrt{A^2m^2 - B^2}} \qquad Y_1 = \frac{-B^2}{\pm\sqrt{A^2m^2 - B^2}}$$

for the co-ordinates of its point of contact.

The above radical value of b is imaginary when $A^2m^2 < B^2$; there are, therefore, directions in which no tangents can be drawn. But when $A^2m^2 > B^2$, two parallel tangents can be drawn, one touching each branch of the curve, as in the figure.

If $A^2m^2 = B^2$, or $m = \pm\frac{B}{A}$, we find $b = 0$, and the equation of the tangent becomes that of one of the asymptotes, X_1 and Y_1 becoming infinite. Hence the asymptotes themselves fulfil the condition of tangency. Since tangents are impossible when m is numerically less than these values, and possible when it is greater, the asymptotes are the *limits of the tangent* as well as of the diameter; and no tangent can be parallel to a diameter.

240. Let the conjugate hyperbola be drawn. Since its equation is $A^2Y^2 - B^2X^2 = A^2B^2$, we can find the co-ordinates of its intersections with the secant line, by making the proper changes of sign in the equation solved in Art. 238. The term added to complete the square will be the same, and changing the sign of the term $-A^2B^2(A^2m^2 - B^2)$ in the numerator of the second member will have no other effect than that of changing the sign of the first two terms in the radical. Therefore the radical in the values of X and Y will become

$$\sqrt{A^2m^2 - B^2 + b^2}.$$

Since the rational parts are unchanged, we shall have the same co-ordinates for the middle point of the chord, and of course the same relation between them. Therefore *parallel chords of conjugate hyperbolas are bisected by the same diameter.*

The condition of tangency to the conjugate hyperbola, found by making the above radical equal zero, is $b = \pm\sqrt{B^2 - A^2m^2}$. Since the quantity under the radical sign is the negative of that in the last Article, tangents to this hyperbola are possible for those values of m which make the tangent to the other impossible. Thus, a tangent to either of these curves is parallel to a diameter of the other.

241. Draw the diameter DD of the conjugate hyperbola parallel to the secant and tangents in the figure, its equation is $Y = mX$. The equation of the diameter P_1P_1 was found to be $Y = m'X$, in which $m' = \frac{B^2}{mA^2}$, therefore the direction ratios of these diameters are connected by the relation,

$$mm' = \frac{B^2}{A^2}.$$

The form of this relation shows, as in the case of the ellipse, that *each* of the diameters $Y = mX$ and $Y = m'X$ bisects chords parallel to the other, and is parallel to tangents at the extremities or vertices of the other. They are called *conjugate diameters* of both of the hyperbolas. Since the axes are rectangular, m and m' are the tangents of the inclinations of conjugate diameters to the transverse axis. Their product being the square of the tangent of the inclination of either asymptote, each asymptote is *conjugate to itself*, or is the limit of a pair of conjugate diameters; one, of those whose inclinations are both acute, the other, of those whose inclinations are both obtuse. In the equilateral hyperbola, $B = A$, therefore $mm' = 1$; in which case the acute angles are complements, the tangent of one being the cotangent of the other.

Equations of the Tangent.

242. We may express the equation of the tangent, $Y = mX \pm \sqrt{A^2m^2 - B^2}$, in terms of the co-ordinates of its point of contact, by the same method that we used for the ellipse in Art. 216. Thus, referring to the values of X_1 and Y_1, in Art. 239, we see that the radical or value of b in the equation of the tangent (whether its sign be positive or negative), equals $-\frac{B^2}{Y_1}$; and that $m = \frac{B^2X_1}{A^2Y_1}$. Making these substitutions and clearing of fractions, we obtain

$$A^2YY_1 - B^2XX_1 = - A^2B^2.$$

To express the tangent at a given point in terms of a single arbitrary constant, instead of the two co-ordinates X_1 and Y_1, we use the auxiliary angle ψ corresponding to the point of contact. Then, by Art. 236, $X_1 = A \sec \psi$ and $Y_1 = B \tan \psi$. Substituting and dividing by AB, we have

$$A \tan \psi \,.\, Y - B \sec \psi \,.\, X = - AB,$$

which represents a tangent whatever the value of ψ.

EXAMPLES.—Show that P_1 satisfies this last equation.

Find by each equation the tangents at the vertices of the transverse axis, and at the extremity of the parameter.

Prove that if an ordinate and a tangent, drawn from the same point of a circle or ellipse having AA′ for an axis, meet the axis in R and T, and an ordinate to the hyperbola be erected at T; then a tangent at the extremity of this ordinate will pass through R.

243. The equation of the tangent to the *conjugate hyperbola* is

$$Y = mX \pm \sqrt{B^2 - A^2m^2},$$

the value of b being found by the condition of tangency in Art. 240. Substituting this value of b in the expressions for the co-ordinates of M, Art. 238, which apply to each curve, we have

$$X_1 = \frac{A^2m}{\pm\sqrt{B^2 - A^2m^2}} \quad \text{and} \quad Y_1 = \frac{B^2}{\pm\sqrt{B^2 - A^2m^2}}.$$

Using these values of X_1 and Y_1, we obtain for the tangent,

$$A^2YY_1 - B^2XX_1 = A^2B^2,$$

which differs from the formula found in the last Article only in the sign of the second member.

To express this tangent in terms of the auxiliary angle ψ, we must put $X_1 = A \tan \psi$, $Y_1 = B \sec \psi$, for these are the co-ordinates of a point on the conjugate hyperbola, by Art. 236. Hence

$$A \sec \psi \,.\, Y - B \tan \psi \,.\, X = AB$$

is the equation of a tangent to $A^2Y^2 - B^2X^2 = A^2B^2$.

EXAMPLES.—Prove that tangents to the conjugate hyperbola, at the points where it is cut by the tangent at the vertex A, pass through the other vertex A′.

Find the tangents to each curve for the values $\psi = 90°$ and $\psi = 270°$. (Multiply the equations by $\cos\psi$, before giving ψ its value.)

244. To express the equation of a tangent in the form $x \cos a + y \sin a = p$, we may use the same method as in Art. 193, for the ellipse. Or, since the inclination of the tangent line is $90° \pm a$, its tangent $m = -\cot a$. Substituting in $Y = mX \pm \sqrt{A^2m^2 - B^2}$, and multiplying each member by $\sin a$, we have (giving p the positive sign)

$$X \cos a + Y \sin a = \sqrt{A^2 \cos^2 a - B^2 \sin^2 a}.$$

Any tangent may be represented by this equation, if we give to a its proper value; parallel tangents having values of a differing by 180°, and therefore equal values of the perpendicular. Thus, $a = 0°$ gives $X = A$, the tangent at the right vertex; $a = 180°$ gives $-X = A$, the tangent at the left vertex. If $A \cos a = \pm B \sin a$, we find zero for the perpendicular; there is therefore a value of a in the first, and another in the second quadrant, for which the tangent passes through the centre. Between these values the tangent is impossible, since the perpendicular becomes imaginary. The opposite values of a in the third and fourth quadrants are also limiting values of a, between which the tangent is impossible. The limiting values may easily be shown to be the inclinations of perpendiculars to the asymptotes.

245. We may now demonstrate properties of the hyperbola, analogous to those of the ellipse in Art. 194. First:

The sum of the squares of perpendiculars from the centre upon perpendicular tangents is constant, and equals $A^2 - B^2$.

For the inclination of CR being a, and that of CR′, $90° + a$,

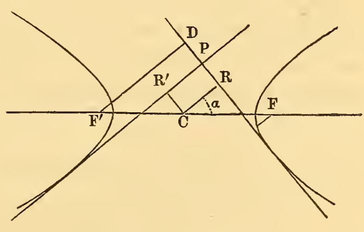
$$CR^2 = A^2 \cos^2 a - B^2 \sin^2 a$$

and

$$CR'^2 = A^2 \sin^2 a - B^2 \cos^2 a;$$

hence

$$CR^2 + CR'^2 = A^2 - B^2.$$



Since CRPR′ is a rectangle, the square of the diagonal is equal to the sum of the squares of two adjacent sides. Therefore $CP^2 = A^2 - B^2$; that is, the distance of P from the centre is constant, and its locus is a circle whose radius is $\sqrt{A^2 - B^2}$.* In the figure, $A > B$, and therefore CP is possible; but if $A = B$, $CP = 0$, and if $A < B$, CP is imaginary. Accordingly, to the *rectangular* hyperbola there is *but one pair of perpendicular tangents*, namely, the asymptotes which intersect at the centre; and to an *obtuse* hyperbola there are *no* perpendicular tangents.

The locus of the foot of a perpendicular from a focus upon a tangent is the circle described on the transverse axis.

For, in the figure, $RD = c \sin \alpha$. But $CD^2 = CR^2 + RD^2$, hence

$$CD^2 = A^2 \cos^2 \alpha + (c^2 - B^2) \sin^2 \alpha = A^2, \quad \text{or} \quad CD = A.$$

The product of the perpendiculars from the foci upon a tangent is constant, and equals $- B^2$.

For the perpendiculars are $c \cos \alpha - \sqrt{A^2 \cos^2 \alpha - B^2 \sin^2 \alpha}$, and $- c \cos \alpha - \sqrt{A^2 \cos^2 \alpha - B^2 \sin^2 \alpha}$ (see Art. 73); and the product of these quantities is $- B^2$. The negative sign of this product is due to the fact that the foci are always on opposite sides of the tangent.

Conjugate Diameters.

246. We found, in Art. 241, the relation

$$mm' = \frac{B^2}{A^2}$$

between the direction ratios of conjugate diameters; that is, between the tangents of the inclinations of diameters to conjugate

* The values of CR^2 for the ellipse and for the hyperbola differ only in the sign of B^2. If for B^2 in the former case, and $- B^2$ in the latter, we substitute $A^2 - c^2$, we have for either curve, $CR^2 = A^2 - c^2 \sin^2 \alpha$. *If another conic have the same foci*, so that the value of c is the same, and if A' represents its transverse or major semi-axis, the value of CR'^2 for a perpendicular line, tangent to the second curve, will be $CR'^2 = A'^2 - c^2 \cos^2 \alpha$. The value of CP will still be constant, and the *locus of the intersection of perpendicular tangents will be the circle whose radius is* $\sqrt{A^2 + A'^2 - c^2}$. If $A' = c$ this second conic reduces to the line FF', and its tangents become lines passing through the foci, which accordingly intersect the tangents to the first conic on the circle whose radius is A.

hyperbolas, of which each is parallel to tangents at the vertices of the other.

Let ψ denote the auxiliary angle of any point on the hyperbola; then its co-ordinates are $X = A \sec\psi$, $Y = B \tan\psi$. The direction ratio of the diameter passing through this point is $m = \frac{Y}{X} = \frac{B \tan\psi}{A \sec\psi}$. Now the co-ordinates of a point on the conjugate hyperbola are $X = A \tan\psi$, $Y = B \sec\psi$; therefore the direction ratio of a diameter to this curve is of the form $\frac{B \sec\psi}{A \tan\psi}$. Supposing the value of ψ to be the same for two points, one on each curve, the product of the direction ratios of diameters drawn to these points will be

$$\frac{B \tan\psi}{A \sec\psi} \times \frac{B \sec\psi}{A \tan\psi} = \frac{B^2}{A^2};$$

therefore the diameters will be conjugate. Hence, *the vertices of conjugate diameters correspond to the same value of* ψ.

247. Let P and P′ be the vertices of a pair of conjugate diameters, we can now express both co-ordinates of each point in terms of ψ. Adding the squares of the co-ordinates to obtain the squares of the distances from the centre, we have

$$CP^2 = A^2 \sec^2\psi + B^2 \tan^2\psi$$

and $$CP'^2 = A^2 \tan^2\psi + B^2 \sec^2\psi.$$

Subtracting, $$CP^2 - CP'^2 = A^2 - B^2,$$

since $\sec^2 - \tan^2 = 1$; that is, *the difference of the squares of conjugate semi-diameters is constant, and equal to* $A^2 - B^2$.

In the equilateral hyperbola, this difference is zero; therefore all its conjugate diameters are equal. When $A > B$, or the transverse axis is greater than the conjugate, every diameter is greater than its conjugate.

248. If for B^2, in the expression for CP'^2, we put its value $c^2 - A^2$, we derive $CP'^2 = c^2 \sec^2\psi - A^2 = e^2x^2 - A^2$, putting Ae for c, and for $A \sec\psi$, x denoting the abscissa of the point P. But $e^2x^2 - A^2$ is the product of the focal distances of P, r and r' in Art. 221. Hence, *the product of the focal distances of the vertex of a*

diameter equals the square of the conjugate semi-diameter. This was also proved for the ellipse in Art. 197.

In the equilateral hyperbola, the conjugate diameters being equal, $CP^2 = rr'$, or the distance of a point on the equilateral hyperbola from the centre is a mean proportional between its distances from the foci.

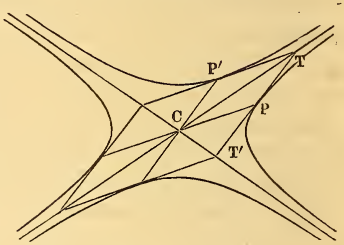
249. The equations of tangents in terms of ψ, which we found in Arts. 242 and 243—namely,

$$A \tan\psi \,.\, Y - B \sec\psi \,.\, X = -AB$$

and

$$A \sec\psi \,.\, Y - B \tan\psi \,.\, X = AB,$$

are the equations of tangents at P and P′ the vertices of conjugate diameters, since the value of ψ for these points is the same. In the figure, P and P′ are so taken that ψ is in the first quadrant.



These tangents intersect the asymptote $AY = BX$ *in the same point.* For, combining $AY = BX$ with the first equation, eliminating successively Y and X, we have for the co-ordinates of intersection,

$$X = \frac{A}{\sec\psi - \tan\psi} \quad \text{and} \quad Y = \frac{B}{\sec\psi - \tan\psi}.$$

Combining the equation of the asymptote with the second equation, we find the same values of X and Y; that is, the same point of intersection.

The four tangents at the vertices of the conjugate diameters evidently form a parallelogram with centre at C. Each side is equal to the diameter to which it is parallel, and is bisected at the point of contact. The above are the equations of two of the sides, and we have seen that they meet one of the asymptotes in the same point. The equations of the other sides of the parallelogram, which are parallel and equally distant from the origin on the other side, will be found by merely changing the sign of the second member. Now find the intersection of the other asymptote, $AY = -BX$, with the tangent at P, and also with the tangent parallel to that at P′. We find *in both cases*

$$X = \frac{A}{\sec\psi + \tan\psi} \qquad \text{and} \qquad Y = \frac{-B}{\sec\psi + \tan\psi};$$

that is, these tangents also meet one of the asymptotes in the same point. Hence the co-ordinates found in this Article are those of T and T′, the angular points of the parallelogram, and the diagonals CT and CT′ are the asymptotes. Therefore, *the asymptotes are the diagonals of the parallelogram formed by tangents at the vertices of conjugate diameters.*

From this it is evident that, if a tangent cut the asymptotes, the intercepted portion is equal to the parallel diameter of the conjugate hyperbola, and *is bisected at the point of contact.*

250. The distances of the points T and T′ from the centre, are the square roots of the sums of the squares of their co-ordinates. Hence we have for the semi-diagonals (since $A^2 + B^2 = c^2$),

$$CT = \frac{c}{\sec\psi - \tan\psi} \qquad \text{and} \qquad CT' = \frac{c}{\sec\psi + \tan\psi}.$$

Therefore $$CT \times CT' = c^2;$$

that is, *the product of the intercepts made by a tangent upon the asymptotes is constant, and equals c^2.*

It follows that the triangle TCT′, formed by a tangent and the asymptotes, is constant in area; for one angle and the product of the including sides is constant. When the point of contact is at the principal vertex, CT and CT′ are each equal to c, and the area of the triangle is AB, the product of the semi-axes (see Fig. Art. 234). Therefore four times this triangle, or *the parallelogram of conjugate diameters, equals the rectangle of the axes.*

Examples.—Find the values of the conjugate semi-diameters, corresponding to $\psi = 0°$, $30°$, $45°$ and $60°$.

Find the value of ψ for which $CP = c$, also that for which $CP' = c$. (Give the values of $\tan\psi$ in each case.)

Find the equations of the diameters CP and CP′ by means of the co-ordinates of P and P′, and show from these equations that they are parallel respectively to the tangents at P′ and P.

Show by the formula for the distance of two points, that $TT' = 2CP'$.

Prove that the transverse semi-axis is a mean proportional between the abscissas of T and T′.

Find the co-ordinates of T directly as the intersection of the tangents of Art. 249.

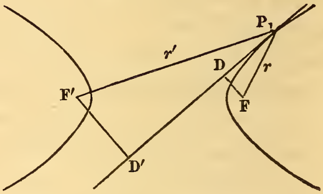
Prove by the values in Arts. 247 and 250, that $CT^2 + CT'^2 = 2(CP^2 + CP'^2)$.

Tangent and Focal Lines.

251. The equations of the lines joining a given point of the hyperbola to the foci may be expressed in terms of the auxiliary angle; and then one of the equations of lines bisecting their angles, found as in Art. 200 for the ellipse, would be that of a tangent line. It may, however, be proved that the tangent makes equal angles with the focal lines, in the following more simple manner.

Let P_1D be a tangent, and let r and r' denote the focal distances of the point of contact P_1. Then by Art. 221,

$$r = eX_1 - A \qquad \text{and} \qquad r' = eX_1 + A.$$



Draw the perpendiculars FD and F′D′ from the foci. The ratios of these perpendiculars to the corresponding focal distances are the sines of the angles at P_1. To find them we must express the perpendiculars in terms of the co-ordinates of P_1. Therefore, using the equation

$$A^2YY_1 - B^2XX_1 + A^2B^2 = 0$$

for the tangent, and the method of Art. 73 for the perpendicular from any point, we have for the foci F, $(Ae, 0)$ and F′, $(-Ae, 0)$,

$$FD = \frac{-AB^2(eX_1 - A)}{\sqrt{A^4Y_1^2 + B^4X_1^2}} \quad \text{and} \quad F'D' = \frac{AB^2(eX_1 + A)}{\sqrt{A^4Y_1^2 + B^4X_1^2}}.$$

These values are of opposite signs because the foci are on opposite sides of the tangent. Dividing their numerical values respectively by r and r', we have the *same value* for $\sin FP_1D$ and $\sin F'P_1D'$;

therefore *the angles* FP_1D *and* $F'P_1D'$ *are equal.* A similar proof may be used in the case of the ellipse.

252. It has been proved, in Art. 245, that the product of the perpendiculars FD and F′D′ is numerically equal to B^2, and in Art. 248, that the product of the focal distances FP_1 and $F'P_1$ is equal to the square of the semi-diameter conjugate to CP_1 or parallel to the tangent. Denoting this semi-diameter by CP′, as in previous Articles, we have

$$\sin FP_1D \sin F'P_1D' = \frac{B^2}{CP'^2}, \qquad \text{or} \qquad \sin FP_1D = \frac{B}{CP'},$$

since these angles are equal. This expression applies also to the ellipse, for the same properties were proved of the focal perpendiculars and distances, in Arts. 194 and 197. Hence, both for the ellipse and for the hyperbola, this expression gives the sine of the angle made at any point of the curve by a tangent with either of the focal distances.* In both cases, the greatest value of the sine of this angle is at the vertices of the major or transverse axis, for which $CP' = B$, its least possible value; the corresponding value of the angle being 90°. In the ellipse the least value of the angle occurs at the vertices of the minor axis, where $CP' = A$, which is its greatest possible value; but in the hyperbola, since CP′ may be increased without limit, the angle may be diminished without limit.

253. The equation of the normal at a given point; that is, the straight line perpendicular to the tangent at the point of contact,

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* The expression for the tangent of this angle in terms of the co-ordinates of the point is of a simple form. Thus, denoting the angle by τ, $\sin\tau = \frac{B}{CP'}$, therefore $\tan\tau = \frac{B}{\sqrt{CP'^2 - B^2}}$. But $\sqrt{CP'^2 - B^2} = c\sin\phi$ for the ellipse, and $= c\tan\psi$ for the hyperbola. (See values in Arts. 196 and 247.) Hence in each case, $\tan\tau = \frac{B^2}{cy}$, since $y = B\sin\phi$ or $B\tan\psi$. This expression gives the acute angle at P when y is positive, and the obtuse value when it is negative. In polar co-ordinates at the focus we have, since y is a perpendicular to the axis, $\tan\tau = \frac{p}{er\sin\theta}$, or substituting the value of r, Art. 229, $\tan\tau = \frac{1 - e\cos\theta}{e\sin\theta}$.

may be found by the formula $y - y_1 = m(x - x_1)$ for a line passing through P_1. To make this line perpendicular to the tangent, we must substitute for m the negative of the reciprocal of the value of m in the equation of the tangent, which is $\frac{B^2x_1}{A^2y_1}$. Hence, we have $y - y_1 = -\frac{A^2y_1}{B^2x_1}(x - x_1)$, or

$$B^2x_1y = -A^2y_1x + c^2x_1y_1,$$

which is the equation of a normal, when P_1 is a point on the curve. Since the value of m in the equation of the tangent to the conjugate hyperbola, Art. 243, is of the same form, the above is the equation of a normal to the conjugate hyperbola when P_1 is a point of that curve. It is easily shown that P_1 always satisfies the equation of the normal.*

EXAMPLES.—Show that the product of the values of FD and F′D′ is $-B^2$ (using the condition that P_1 satisfies the equation of the hyperbola).

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* Since the normal is perpendicular to the tangent, it bisects, in the case of the hyperbola, the *exterior angle* of the focal lines. Hence, if an ellipse and an hyperbola have the same foci, the tangents to the ellipse at the intersections are normal to the hyperbola.

Let A and B denote the semi-axes of the ellipse, and A′, B′ those of the hyperbola. Then $A^2y_1y + B^2x_1x = A^2B^2$ and $B'^2x_1y + A'^2y_1x = c^2x_1y_1$, are equations of the same line, tangent to the ellipse and normal to the hyperbola at P_1. Hence the *ratios* of the coefficients and absolute terms in these equations (that is, the values of the intercepts and direction ratio as determined by them) are equal. Equating the values of x_0, and those of y_0, we obtain $x_1 = \pm\frac{AA'}{c}$ and $y_1 = \pm\frac{BB'}{c}$, for the co-ordinates of intersection. Equating the direction ratios (supposing x_1 and y_1 positive), $\frac{y_1}{x_1} = \frac{BB'}{AA'}$. But $\frac{y_1}{x_1} = \frac{B\sin\phi}{A\cos\phi} = \frac{B'\tan\psi}{A'\sec\psi}$. Substituting these values successively, we find $\tan\phi = \frac{B'}{A'}$, and $\sin\psi = \frac{B}{A}$. Hence, the eccentric angle of the point in which an ellipse is cut by a confocal hyperbola is the inclination of the asymptotes of the hyperbola; and the auxiliary angle in the hyperbola corresponding to the same point is the inclination of a line joining the focus with the vertex of the minor axis of the ellipse.

Prove that the parts of the normal intercepted between the axes, between P_1 and the axis of X and between P_1 and the axis of Y have the same ratios as c^2, B^2 and A^2. (This property belongs also to the normal to the ellipse.)

Find the values of A^2 and B^2 for the conic section to which $y - y_1 = m(x - x_1)$ is normal at P_1. (Since $m = -\frac{A^2 y_1}{B^2 x_1}$ in the normal, $\frac{B^2}{A^2}$ may be expressed in terms of m, x_1 and y_1; and then the value of A^2 may be found by an equation of condition expressing that the curve passes through P_1. If $\frac{B^2}{A^2}$ as thus found would be negative, we must use the normal of Art. 201, and determine an ellipse.)

Determine in this way the conic normal to $y = 2x + 2$ at $(1, 4)$ and verify the result by finding the normal at the given point. (This curve being an ellipse in which $A^2 < B^2$, we must give the negative sign to c^2, which was put for $A^2 - B^2$ in the equation of the normal, Art. 201.)

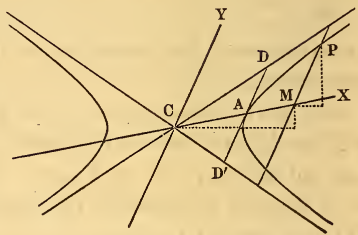
Determine the conic normal to $y + 2x = 5$ at $(2, 1)$. (The value found for A^2 will be zero, therefore the equation reduces to that of the asymptotes determined by the value found for $\frac{B^2}{A^2}$.)

Hyperbola Referred to Conjugate Diameters.

254. The equation of the hyperbola as referred to a pair of conjugate diameters may be found by the method that we used for the ellipse in Art. 202. Thus, adding the squares of the co-ordinates of M, or rational parts of the values of X and Y in Art. 238, for the square of the oblique abscissa CM; and adding the squares of the radical parts for PM^2, we have

$$X^2 = \frac{b^2(A^4m^2 + B^4)}{(A^2m^2 - B^2)^2}, \qquad Y^2 = \frac{(1 + m^2)A^2B^2(B^2 - A^2m^2 + b^2)}{(A^2m^2 - B^2)^2}.$$

In these equations m is constant, and we have to eliminate b^2. To simplify the result we introduce as new constants the values of the conjugate semi-diameters. Let A denote that which is measured on the new axis of X, which we will suppose to cut the hyperbola, as in the figure. Then $A^2m^2 - B^2$ is a positive quantity, and if we give this value to b^2, the line PP will become a tangent as explained in Art. 239, and accordingly the value of Y^2 becomes zero. The corresponding value of X^2 is A^2; therefore



$$A^2 = \frac{A^4m^2 + B^4}{A^2m^2 - B^2}.$$

Since the curve does not intersect the new axis of Y, $b = 0$, which makes $X^2 = 0$, will be found to make Y^2 negative. Let B denote the conjugate semi-diameter, or intercept of the conjugate hyperbola on the axis of Y. Now in Art. 240, we found that the values of X and Y for the conjugate hyperbola differ from those in Art. 238 only in the radical, which becomes $\sqrt{A^2m^2 - B^2 + b^2}$. Hence, for the conjugate hyperbola, the value of X^2 is the same as that above, but

$$Y^2 = \frac{(1 + m^2)A^2B^2\,(A^2m^2 - B^2 + b^2)}{(A^2m^2 - B^2)^2}.$$

From this we derive, by making $b = 0$,

$$B^2 = \frac{(1 + m^2)\,A^2B^2}{A^2m^2 - B^2}.$$

Dividing X^2 by A^2, and the *first* value of Y^2 by B^2, we derive

$$\frac{X^2}{A^2} = \frac{b^2}{A^2m^2 - B^2} \qquad \text{and} \qquad \frac{Y^2}{B^2} = \frac{B^2 - A^2m^2 + b^2}{A^2m^2 - B^2}.$$

Hence $\dfrac{X^2}{A^2} - \dfrac{Y^2}{B^2} = 1$ or $A^2Y^2 - B^2X^2 = - A^2B^2$,

an equation of the same form as the original rectangular equation.

Dividing the *second* value of Y^2 by B^2, we obtain in the same way for the equation of the *conjugate* hyperbola,

$$\frac{Y^2}{B^2} - \frac{X^2}{A^2} = 1 \qquad \text{or} \qquad A^2Y^2 - B^2X^2 = A^2B^2,$$

which is also of the same form as the rectangular equation of the same curve.

255. If the value of $A^2m^2 - B^2$ had been negative, the new axis of X would have cut the conjugate hyperbola, and that of Y would have cut the curve of the figure. The values of A^2 and B^2 would be in form the negatives of the values of the last Article; that of A^2 being found by making $b^2 = B^2 - A^2m^2$, which makes the line PP tangent to the conjugate hyperbola, and that of B^2 being found by means of the *first* value of Y^2. Hence the equation of the hyperbola in the figure would take the second of the above forms, and that of its conjugate would take the first form.

Therefore, dropping the distinction between oblique and rectangular co-ordinates, we may regard $A^2Y^2 - B^2X^2 = - A^2B^2$ and $A^2Y^2 - B^2X^2 = A^2B^2$ as the equations of two conjugate hyperbolas, of which the axes are a pair of conjugate diameters. Since conjugate diameters of a given hyperbola may be found making any given angle with each other, either of these equations may be made to represent an hyperbola of any shape, by giving proper values to A and B, whatever be the inclination of the co-ordinate axes.*

256. If on the tangent at the vertex A we lay off AD and AD′, each equal to the conjugate semi-diameter B, and join CD, CD′, the oblique equations of these lines will be

$$AY = BX \qquad \text{and} \qquad AY = -BX,$$

for they pass through the origin and the points (A, B) and $(A, -B)$ respectively. But these lines are evidently the diagonals of the parallelogram constructed, as in Art. 249, on a pair of conjugate diameters, hence they are the asymptotes. These equations are also of the same form as the rectangular equations of the same lines, and therefore the compound equation

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* This is not true of the corresponding equation of the ellipse, since there is a limit to the obliquity of its conjugate diameters. See Note to Art. 206. The general central equation $Ax^2 + Cy^2 + F = 0$ cannot represent a conic having an eccentricity less than that of the ellipse making equal intercepts on the axes.

$$A^2Y^2 - B^2X^2 = 0$$

represents both asymptotes of the curves $A^2Y^2 - B^2X^2 = \mp A^2B^2$, whether the axes are rectangular or oblique. When $A = B$, the equations of the asymptotes reduce to $Y = X$ and $Y = -X$, which are in all cases the equations of lines bisecting the angles between the co-ordinate axes. Therefore when $A = B$, the asymptotes are at right angles and the hyperbola is *rectangular*. But when A and B are unequal, the asymptotes are oblique to each other; and the hyperbola is *acute* when the transverse semi-diameter (that which is intercepted by the curve) is *greater* than its conjugate; for by Art. 247, the transverse semi-axis will then be greater than the conjugate.

257. When the hyperbola and its conjugate are both drawn, or when the equation is given, so that the values of A and B are known, the asymptotes may be drawn, and then the axes of the curve may be drawn bisecting their angles. But when the curve and its centre are given, the axes may be constructed geometrically by the method pointed out for the ellipse in Art. 203; for the property of supplementary chords there proved evidently extends to the hyperbola also. If the centre is not given, it may be found geometrically, by drawing a diameter bisecting parallel chords, and then bisecting the diameter, or if one branch only of the curve is given, by finding the intersection of two diameters.

258. Hyperbolas having their axes proportional have the same eccentricity, and are said to be *similar*. As in the case of similar ellipses, the value of r^2, Art. 226, shows that if their transverse axes are parallel, all the parallel diameters will have the same ratio.

Since the inclination of the asymptotes determines the ratio of the axes, *hyperbolas having the same transverse axis and the same asymptotes are similar*, and their conjugate hyperbolas are also similar.

Let $n = \frac{A}{B}$, then the central equation of any hyperbola may be put in one of the forms,

$$X^2 - n^2Y^2 = \pm A^2,$$

in which n determines the asymptotes, which are the lines $X = nY$

and $X = -nY$. Equations of this form, in which the value of n is the same, therefore represent hyperbolas having *the same asymptotes.* If the sign of the second member is the same in each, they are similar, being situated in the same angles. But if the second members have opposite signs, they are situated in the supplemental angles of the asymptotes; one being acute and the other obtuse, except when $n = 1$, in which case both are rectangular.

If the value of n is fixed, while that of the second member varies, the asymptotes will be fixed. Then, as the second member is diminished, the hyperbola will approach nearer and nearer to the asymptote. When it becomes zero, we have the equation $X^2 - n^2Y^2 = 0$, equivalent to $X = \pm nY$, the equations of the two asymptotes. Therefore the hyperbola is said to *vanish into a pair of straight lines.*

AXES PARALLEL TO CONJUGATE DIAMETERS.

259. The equation of an hyperbola whose centre is at P′ is found by substituting for X and Y, in the central equation, their equivalents $x - x'$ and $y - y'$. Hence

$$A^2(y - y')^2 - B^2(x - x')^2 = \mp A^2B^2,$$

and

$$(x - x')^2 - n^2(y - y')^2 = \pm A^2,$$

are the equations of an hyperbola having a pair of conjugate diameters parallel to the co-ordinate axes. The first is expressed in terms of the lengths of the semi-diameters, the second in terms of a ratio n, which determines the direction of the asymptotes, and a quantity which determines the size of the curve, or its distance from its asymptotes. If we put zero in place of the second member in either of these equations, we produce the equation of the asymptotes. Expanding the last equation, we have

$$x^2 - n^2y^2 - 2x'x + 2n^2y'y + x'^2 - n^2y'^2 \mp A^2 = 0.$$

From this we see that the general equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

represents *an hyperbola, when* A *and* C, *the coefficients of* x^2 *and* y^2, *have opposite signs.*

260. This is the general equation of the conic section having a pair of conjugate diameters parallel to the co-ordinate axes, including, as shown in Art. 209, the parabola having its axis parallel to either of the co-ordinate axes. A conic fulfilling this condition may be found, which shall pass through *four* given points; for the *ratios* of the five coefficients may be determined by four equations of condition. Thus, we may assume the equation in the form $x^2 + cy^2 + dx + ey + f = 0$; and then by substituting for x and y the co-ordinates of each of the given points successively, we shall have four equations by which to determine c, d, e and f. It may happen that these equations will be found incompatible; this will occur when the conic is a parabola of the form $Cy^2 + Dx + Ey + F = 0$, in which case the proper coefficient of x^2 being zero, the equation cannot be expressed in the assumed form. If the four points are in one straight line, we shall find $A = 0$ and $C = 0$.

261. To determine the centre and semi-diameters of the curve from the values of c, d, e and f, we may use the equations of Art. 208, prepared for the ellipse. The only difference is, that c (being negative) is the value of $-n^2$, in the hyperbola, and that $\frac{1}{4}\left(d^2 + \frac{e^2}{c}\right) - f$ may be the value of either $\pm A^2$. Therefore the equation will represent an hyperbola, whether the value of this quantity be positive or negative; but if it be zero, the equation will represent a pair of straight lines. When this quantity equals zero in the equation of an ellipse, the curve vanishes into a single point, as shown in Art. 210. To find the condition that an equation of the above general form shall represent a pair of straight lines or a single point, substitute in $\frac{1}{4}\left(d^2 + \frac{e^2}{c}\right) - f = 0$, the ratios $\frac{D}{A}$ for d, $\frac{E}{A}$ for e, etc. The result is

$$AE^2 + CD^2 - 4ACF = 0.$$

If we suppose C to be positive, the sign of the expression in the first member has not been changed by clearing of fractions, which was effected by multiplying by $4A^2C$. Therefore *making* C *positive* in a given equation, this expression will be *positive* for a *real* ellipse or an hyperbola in the first form; that is, one which *cuts the* *diameter parallel to the axis of* X. If the expression is negative, the ellipse is imaginary, or the hyperbola is of the second form; that is, the equation may be reduced to one of the forms $(x - x')^2 \pm n^2(y - y')^2 = - A^2$.

EXAMPLES.—Determine the form of $x^2 - 2y^2 - 4x + 4y = 0$; of $x^2 - 2y^2 - 4x + 4y + 10 = 0$; of $x^2 + 2y^2 + 4x - 4y + 6 = 0$.

262. If $A = 0$, the equation generally represents a parabola; but we found in Art. 156 that if $D = 0$ it represents a pair of straight lines. Now $A = 0$ and $D = 0$ will satisfy the condition of the last Article independently of the value of F. In like manner, $C = 0$ and $E = 0$, which makes the equation represent lines parallel to the axis of Y, satisfies the condition. It is also satisfied by $A = 0$ and $C = 0$, which reduces the equation to the first degree. Hence the above is the general condition that $Ax^2 + Cy^2 + Dx + Ey + F = 0$ shall represent straight lines.

If in an equation representing an hyperbola we suppose the value of F to vary, the co-ordinates of the centre and the direction of the asymptotes will not be changed. Therefore, if in such an equation we give to F a value determined by the condition for straight lines, we shall obtain the equation of the asymptotes. Thus, given the hyperbola $x^2 - 2y^2 - 4x + 4y + 10 = 0$, we substitute the values of all the coefficients but F in the condition. This gives $16 - 32 + 8F = 0$, hence $F = 2$, and $x^2 - 2y^2 - 4x + 4y + 2 = 0$ is the equation required.

263. The co-ordinates of the centre, in Art. 208, become by substituting their general values for c, d and e,

$$x' = - \frac{D}{2A} \qquad \text{and} \qquad y' = - \frac{E}{2C}.$$

From these values it is evident that if $D = 0$ the centre is on the axis of Y, and if $E = 0$ it is on the axis of X. If both $D = 0$ and $E = 0$, it is at the origin. But if $A = 0$, the value of x' becomes infinite; and if $C = 0$, the value of y' becomes infinite. Therefore the centre of a parabola is said to be at an *infinite distance*. If $A = 0$ and $D = 0$, x' takes the indeterminate form; therefore any point which satisfies $y = - \frac{E}{2C}$, that is, any point on the line $2Cy + E = 0$, may be regarded as the centre of $Cy^2 +$

$Ey + F = 0$, which represents parallel lines. The diameters of the parabola are all parallel or intersect at infinity; but for parallel lines, they are all coincident.

The centre of a pair of intersecting straight lines is their point of intersection, and that of an infinitesimal ellipse is the single point which satisfies the equation. Therefore the condition that the equation shall be satisfied by the centre is equivalent to the condition of Art. 261, as will be found by substituting the values of x' and y' for x and y in the general equation.

264. The equation of the asymptotes of the hyperbolas of Art. 259, is $(x - x')^2 - n^2(y - y')^2 = 0$. Hence they are the two lines $x - x' = \pm n(y - y')$, in which $n = \sqrt{-\frac{C}{A}}$. If we let $m = \sqrt{-\frac{A}{C}}$, these equations may also be written in the form $y - y' = \pm m(x - x')$. By this value of m, and the values of x' and y', in the last Article, we may form the equations of the asymptotes separately. Thus, given $x^2 - 2y^2 - 4x + 4y + 10 = 0$, we find $x' = 2, y' = 1, m = \sqrt{\frac{1}{2}}$, hence the asymptotes are $y - 1 = \pm\sqrt{\frac{1}{2}}(x - 2)$. The compound equation formed from these by the method of Art. 81, will be found identical with that found for the same curve in Art. 262.

Since for an ellipse A and C have the same sign, the above value of m will be imaginary; therefore the ellipse is said to have *imaginary* asymptotes. The compound equation will then be satisfied by a single point, namely the centre. We have hitherto called this the equation of an *infinitesimal ellipse*, because it is the vanishing case of an ellipse of given shape; we may now consider it the equation of two *imaginary straight lines* passing through the centre, whose imaginary direction ratios determine the direction of the axes and the shape of the ellipse.

For the parabola in which $A = 0$, we find the direction ratio $m = \pm 0$, which shows that the *directions* of both asymptotes coincide with that of the axis of X, but since the parabola has no centre, no asymptotes can be found. The directions of asymptotes are those of straight lines which cut the conic in only one point. For the hyperbola there are two such directions, for the parabola but one, and none at all for the ellipse.

Results of Transformation.

265. If the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ be transformed by changing the direction of the axes, the new equation will in general contain a term involving xy, the product of the variables. Since the formulæ of Case IV., Art. 90, give values of x and y containing no absolute term, it is evident that the coefficients A and C will appear only in the terms of the second degree of the transformed equation. Now the shape of the curve and directions of the asymptotes and axes depend only upon A and C; they must, therefore, in every equation representing a conic section, depend only upon the terms of the second degree. We cannot, however, by these terms alone ascertain whether the equation represents a pair of real or of imaginary straight lines, or distinguish between the equations of real and imaginary ellipses. In the next Chapter we shall discuss the general equation of the conic section containing the term xy; the fact that that term is wanting in the present equation expresses the condition that the diameters parallel to the co-ordinate axes are conjugate.

266. Change of origin without change in the direction of the axes does not affect the terms of the second degree, upon which we may consider the *direction* and *shape* of the curve to depend. But it does affect the terms of the first degree, upon which the co-ordinates of the centre, or the *two elements of the position* of the curve, depend.

The absolute term of the equation is not altered by changing the direction of the axes; but when the origin is moved to P′, the new value of F, which we denote by F′, is found to be

$$F' = Ax'^2 + Cy'^2 + Dx' + Ey' + F,$$

that is, the result of substituting the co-ordinates of the new origin in the first member of the given equation. (See Art. 100.)

This quantity becomes zero for a point on the curve, in which case we know that the absolute term should be zero.

267. Supposing the axes to be rectangular, and transforming to polar co-ordinates, we have

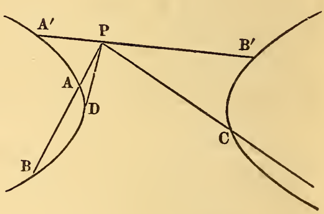
$$(A \cos^2 \theta + C \sin^2 \theta)\, r^2 + (D \cos \theta + E \sin \theta)\, r + F = 0,$$

in which the initial line is parallel to one or other of the principal axes of the curve. This equation gives in general two values of r corresponding to any value of θ. Now the product of the roots of a quadratic equation is the absolute term divided by the coefficient of the highest power. Therefore the product of the two values of r or segments of a chord passing through the pole is

$$\frac{\mathrm{F}}{\mathrm{A}\cos^2\theta + \mathrm{C}\sin^2\theta},$$

where θ is the inclination of the chord to the axis of X. Since, by the last Article, A and C are unchanged by transformation to a new origin, the products of the segments of parallel chords passing through different points *are proportional to the values of* F *for those points;* that is, to the results obtained by substituting the coordinates of the points in the first member of the equation.

268. The above expression is the value of PA $\times$ PB, if F is the absolute term corresponding to the point P in the figure. In general this is a varying quantity. But when A $=$ C (which makes the curve a circle, since the axes are rectangular), it becomes constant, as shown in Art. 128. If P be taken at the focus, PA and PB will be the two values of r in Art. 229, whose product $\dfrac{p^2}{1 - e^2\cos^2\theta}$ is proportional to the focal chord, Art. 232. Therefore the products of the segments of chords passing through the same point in different directions *are proportional to the parallel focal chords.*



If A and C have the same sign, it is evident that the expression for PA $\times$ PB cannot be made to change sign by varying the value of θ, being always *negative* for a point *within* the ellipse, because the segments are measured in opposite directions from P, and *positive* for a point *without* the curve, because PA and PB are measured in the same direction. But when the curve is an hyperbola, the product will change sign, passing through the value infinity, when the line is parallel to an asymptote, as at PC. Thus, for a point *without* the curve (that is, a point from which a tangent may be drawn, as P, in the figure), it has *the same sign as the parallel focal chord* which we found, in Art. 232, to be positive when it cuts a single branch, and negative when it cuts both branches, or is parallel to a diameter of the curve. For a point within either branch of the curve, the reverse would take place, the value of F changing sign as the point passes across the curve.

269. Since the value of F, for a given point, may be computed by the formula of Art. 266, we have the means of finding the value of this product for a given point and given value of θ, without performing the transformation, when the axes are rectangular in the given equation. The same method will of course give the square of a tangent, as PD in the figure.

The combined equation of Art. 213 may now be regarded as expressing that the value of $PA \times PB$ for one of the given ellipses, equals $-k$ times its value for the other, *for the same value of* θ. For the expressions combined are the values of F for the point P, and the values of A and C in the denominators are the same. The straight line $(d - d')x + (e - e')y + f - f' = 0$ is therefore the locus of the point P, when these products are equal, and it bisects the common tangents to the two conics.

Intersections of Conics.

270. If we eliminate one of the variables between two equations of the second degree, the result will be an equation of the fourth degree, containing the other variable. Although such an equation cannot be solved by the rules of Common Algebra, we know that in some cases it can be satisfied by *four* real values of the unknown quantity. Therefore the loci of two equations of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

may intersect in four points. We shall use the term *conic* to denote the locus of this equation, which as we have seen may be a curve or a pair of straight lines. Let S represent the polynomial which constitutes the first member of the equation; then $S = 0$ is the equation of a conic. Let $S' = 0$ represent another equation of the same form, then the combined equation,

$$S + kS' = 0,$$

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will be an equation of the same form; that is, it will consist of terms containing the squares and first powers of the variables, and an absolute term. But it is satisfied by those values of x and y which satisfy both $S = 0$ and $S' = 0$; therefore it represents a conic passing through all the intersections of the conics $S = 0$ and $S' = 0$.

271. From the form of the equations combined, we know that the given conics have each a pair of conjugate diameters parallel to the co-ordinate axes. Since $S + kS' = 0$ is an equation of the same form, the conic it represents fulfils the same condition, whatever be the value of k. Therefore, if $S = 0$ and $S' = 0$ intersect in four points, $S + kS' = 0$ represents a conic of which the direction of a pair of conjugate diameters is fixed, and which passes through four fixed points; k remaining arbitrary, so that another condition may be fulfilled.* For instance, we may determine k so as to make the term containing x^2 disappear, which will make the conic a parabola of the form $Cy^2 + Dx + Ey + F = 0$; that is, one whose axis is parallel to the axis of X. Or we can make the term containing y^2 disappear, and thus determine a parabola with axis parallel to the axis of Y.

We may also determine k by the condition that the coefficients of x^2 and y^2 in $S + kS' = 0$ shall be equal. This will make the conic an ellipse, in which the diameters parallel to the co-ordinate axes shall be the equal conjugate pair, but will not make it a circle unless the axes are *rectangular*. Again, we can make these coeffi-

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* It would seem from this that a conic of this form, which already fulfils one condition, might be found passing through *five* points, whereas we saw in Art. 260, that it can only be made to pass through four given points. The explanation is that the four *fixed* points are not four *given* points; and that passing through them really constitutes but *three* conditions, because any three of them imply the fourth. For let A, B and C be three given points and $S = 0$, $S' = 0$ the equations of two conics of the above form passing through them. Then $S + kS' = 0$ represents *all* the conics of this form, which pass through A, B and C; because, by Art. 260, there is generally *but one* passing through these three and a given fourth point, and if we properly determine k, $S + kS' = 0$ will become its equation. But $S = 0$ and $S' = 0$ evidently intersect in a certain fourth point, and $S + kS' = 0$ always passes through that point. Hence also, four points and the condition implied in the form of the equation do not *always* determine a conic.

cients equal but of opposite signs, which will always determine an equilateral hyperbola. See Art. 256.

272. If A, C, D, E and F are the coefficients in $S = 0$, and A′, C′, D′, E′ and F′, in $S' = 0$, then $A + kA'$, $C + kC'$, etc. will be the corresponding coefficients in $S + kS' = 0$. Therefore the conditions determining k, in the last Article, give rise to equations of the first degree. But if we substitute these coefficients in the condition of Art. 261, the result is an equation of the third degree, to determine k so that the equation $S + kS' = 0$ shall represent either two straight lines or a single point. Now an equation of the third degree may be satisfied by three values of k; accordingly, when $S = 0$ and $S' = 0$ meet in four points A, B, C and D, the pairs of straight lines, AB and CD, AC and BD, AD and BC, constitute three cases in which the conic $S + kS' = 0$ satisfies the condition.

If $S = 0$ and $S' = 0$ intersect in only two points, the equation of the third degree will be satisfied by only one value of k; and that value will make $S + kS' = 0$ a pair of straight lines, one of which passes through the two points of intersection (or is a common chord), and the other fails to meet either of the given conics. But if its equation were combined with each of the given equations, there would result the same imaginary values of x and y; therefore the conics are said in this case to meet in *two real and two imaginary points*. If the conics merely touch each other in a single point, the common chord becomes a common tangent, and the two real points become *coincident*. If the conics do not meet at all, the straight lines are said to meet the curves in *four imaginary points*.*

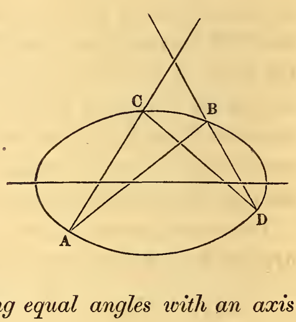
273. When $A : C :: A' : C'$; that is, when the coefficients of x^2 and y^2, in $S = 0$ and $S' = 0$, are proportional; the value $k = -\frac{A}{A'} = -\frac{C}{C'}$ will reduce $S + kS' = 0$ to an equation of the first degree.† But for other values of k it will represent a series

* In this case, as well as when the intersection is in four real points, the cubic equation would have three real roots; but two of the roots would now give infinitesimal ellipses or vanishing points.

† This value of k will of course satisfy the cubic equation, but one of the pair of lines is the *line at infinity*, which intersects hyperbolas with parallel asymptotes in two *real* points at infinity—namely, the intersections of the

of similar and parallel ellipses, like the equation of Art. 213, or a series of hyperbolas with parallel asymptotes, as that equation would if the negative sign were given to n^2. In the latter case, it may be proved, as in Art. 214 for the ellipse, that the common chord $(d - d')x + (e - e')y + f - f' = 0$ is parallel to the diameters conjugate to the common diameter, or line joining the centres of the given conics. Now if the given hyperbolas have a common asymptote, this line must be parallel to it; for the line joining the centres will be the asymptote, and an asymptote is conjugate to itself. But a line parallel to an asymptote cuts an hyperbola in only one point; therefore, in this case, the hyperbolas will cut in a *single* point, and $S + kS' = 0$ will represent a series of hyperbolas with a common asymptote.*

274. Let the co-ordinate axes be *rectangular;* then $S + kS' = 0$ represents a series of conics whose axes are parallel; for the axes are the only rectangular pair of conjugate diameters. Let $S = 0$ and $S' = 0$ intersect in the four points A, B, C and D. Now when $S + kS' = 0$ becomes the pair of straight lines AB and CD, its axes become the lines bisecting the angles between these lines. Hence, the lines which bisect the angles between the chords AB and CD are parallel to the axes of $S = 0$, or these chords *are equally inclined in opposite directions to an axis of the conic.* For the same reason, AC and BD are equally inclined to the same axis. Hence, *if two intersecting chords,* AB *and* CD, *be drawn, making equal angles with an axis*



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parallel asymptotes. See Art. 43. But it intersects a parabola in two *coincident* points, and an ellipse in two *imaginary* points, because the *directions* of the asymptotes become, in these cases, respectively coincident and imaginary, as shown in Art. 264. If the conics are concentric also, both of the straight lines and all four of the intersections are at infinity. Compare Art. 133 and Note.

* Since an asymptote is a *tangent at infinity,* these hyperbolas are said to have three points at infinity, two of which are *coincident.* Concentric conics have two pairs of coincident points at infinity.

of a conic, the chords AC *and* BD *will also be equally inclined to the axis.* It follows that the angles ACD and ABD will be equal when the chords are thus drawn. Since the conic may be a circle, of which any diameter may be taken as an axis, we have, as a special case of this theorem, the well known property of the equality of all the angles in the same segment of a circle.

Reciprocal Polars.

275. The form of the equation of the hyperbola, as referred to any pair of conjugate diameters, being the same as that of the rectangular equation used in Art. 237, the condition of tangency and equation of the tangent are of the same form in this case as when the curve is referred to its axes. Hence the equations of Arts. 242 and 243, for a tangent in terms of the co-ordinates of its point of contact, apply when the axes are oblique.

In general, the equations

$$A^2YY_1 - B^2XX_1 = \mp A^2B^2$$

represent straight lines which are called *polars* of P_1, with respect to the hyperbolas $A^2Y^2 - B^2X^2 = \mp A^2B^2$. It may be shown, as in Art. 217 for the ellipse, that if P_2 is on the polar of P_1, then P_1 is on the polar of P_2; that is, P_1 and P_2 are *reciprocally polar.* Again it may be shown, as in Art. 218, that the polar of a point on any diameter is parallel to the conjugate diameter, and cuts it at a distance from the centre which forms a third proportional to the distance of the point and the semi-diameter.

The more general equation of Art. 219 evidently applies to the hyperbola as well as to the ellipse, and in general,

$$Axx_1 + Cyy_1 + \tfrac{1}{2}D\,(x + x_1) + \tfrac{1}{2}E\,(y + y_1) + F = 0$$

is the formula for the polar of P_1, with respect to the conic

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

276. Since the formula for the polar contains two arbitrary constants, it may in general be made the equation of any given line, by giving proper values to x_1 and y_1. The point of which a given line is the polar is called its *pole.* To find the pole of a given line, that is, the proper values of x_1 and y_1, select two points of the line,

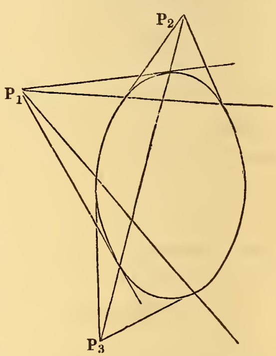
and find the intersection of their polars. If we denote the two points by P_2 and P_3, the algebraic process will evidently be the same as that of finding P_1 by two equations of condition, expressing that its polar passes through P_2 and P_3. This is a consequence of the reciprocal property above referred to.

277. This reciprocal property, and the fact that the polar of a point on the curve is a tangent at that point, furnish the following geometrical definitions of the polars of given points and the poles of given lines.

If tangents be drawn at the points in which a secant line cuts the curve, their intersection is the pole of the secant. For these tangents are the polars of two points on the secant. Thus P_1, in the figure, is the pole of P_2P_3. Every secant not passing through the centre has a pole, but a diameter has no pole, because the tangents at its extremities are parallel.

If tangents be drawn through a given point without the curve, the line joining the points of contact is the polar of the given point. For these points are the poles of two lines passing through the given point. If the curve is an hyperbola and the given point the centre, the tangents become asymptotes, and the points of contact are at an infinite distance.

Now if the polar of P_1 be constructed for any conic, it will contain the poles of all lines passing through P_1, for such points are reciprocally polar to P_1. Hence we derive the following general property of the conic sections:



If tangents and secants be drawn through any point, the intersections of pairs of tangents, drawn at the points where each secant cuts the curve, will lie in a straight line with the first points of tangency.

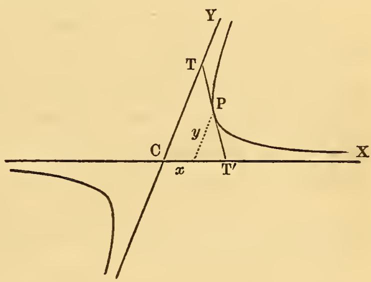
Thus, the polar of P_1 is the locus of all points which can be constructed in the same manner as P_2 and P_3 in the figure. If the point P_1 be taken within the conic, so that tangents cannot be drawn through it, its polar may be constructed by joining the poles of any two lines drawn through it.

Therefore, *the locus of the intersection of tangents at the extremities of a chord drawn through a given point is a straight line*, and this straight line is the *polar* of the given point.

If the chord or secant drawn through P_1 is a diameter, the tangents are parallel to each other, and therefore parallel to the polar. From this it is evident that the polar is parallel to the diameter conjugate to that passing through the pole.

Hyperbola Referred to its Asymptotes.

278. It is proved in Art. 249 that the portion of the tangent intercepted between the asymptotes is bisected at the point of contact, and in Art. 250, that the product of the intercepts on the asymptotes equals c^2. These properties enable us to find the equation of the hyperbola as referred to its asymptotes. For let P be any point of an hyperbola whose asymptotes are the co-ordinate axes CX and CY. Draw the tangent at P, cutting the axes in T and T′, and draw the ordinate of P. Then since $PT' = \frac{1}{2}TT'$ by similar triangles $y = \frac{1}{2}CT$ and $x = \frac{1}{2}CT'$. But $CT \times CT' = c^2$; therefore



$$xy = \tfrac{1}{4}c^2,$$

or the product of the co-ordinates equals one-fourth of the square of the distance from the centre to the focus. The form of this equation shows that the curve does not cut either of the axes, for neither x nor y can be made equal zero. Since the product of the co-ordinates is positive, the equation is also satisfied by points of which the co-ordinates are both negative, which constitute the other branch of the curve.

279. The equation of the tangent TT′ is most readily expressed in terms of the co-ordinates of its point of contact. For this purpose denote the co-ordinates of P, in the figure, by x_1 and y_1. Then $CT' = 2x_1$ and $CT = 2y_1$. But these are the intercepts of the tangent; therefore, using the formula $\frac{x}{a} + \frac{y}{b} = 1$, we have $\frac{x}{2x_1} + \frac{y}{2y_1} = 1$

or $xy_1 + yx_1 = 2x_1y_1$. But since P_1 satisfies the equation of the curve, $x_1y_1 = \frac{1}{4}c^2$, and the equation of the tangent may be written in the form

$$xy_1 + yx_1 = \tfrac{1}{2}c^2.$$

This is the equation of a tangent, only when P_1 is a point of the curve, since otherwise the last substitution could not be made.

The equation of the diameter passing through P_1 is $yx_1 = xy_1$, of which the direction ratio is $\frac{y_1}{x_1}$. That of the tangent is $-\frac{y_1}{x_1}$. Therefore the direction ratios of conjugate diameters as referred to the asymptotes are the *negatives* one of the other.

280. The conjugate hyperbola has the same asymptotes and the same value of c. But one of the co-ordinates of any point of it would be negative and the other positive; hence its equation is

$$xy = -\tfrac{1}{4}c^2.$$

Therefore the equations of conjugate hyperbolas, thus referred, differ in the same way as when they are referred to conjugate diameters—namely, in the sign of the absolute term. Moreover, if we put zero in place of the absolute term, we shall have the equation of the asymptotes. For the equations of the axes are $x = 0$ and $y = 0$, and their compound equation is $xy = 0$.

The shape of each of the hyperbolas $xy = \pm\frac{1}{4}c^2$ is determined solely by the inclination of the co-ordinate axes. They are both rectangular or equilateral if the axes are rectangular.

The axes of the curves are the lines $x = y$ and $x = -y$, which are perpendicular, because they bisect the angles between the co-ordinate axes, whatever be their inclination.

281. If the equation of a straight line, $y = mx + b$, be combined with $xy = \frac{1}{4}c^2$, we shall find for the intersections,

$$y = \tfrac{1}{2}b \pm \tfrac{1}{2}\sqrt{c^2m + b^2}, \qquad x = -\tfrac{1}{2}\frac{b}{m} \pm \frac{1}{2m}\sqrt{c^2m + b^2}.$$

From these values may be derived the equation

$$y = mx \pm c\sqrt{-m},$$

for the tangent having a given direction, and the equation $y = -mx$, for a diameter bisecting chords parallel to $y = mx$.

A line of the form $x = a$, or one of the form $y = b$, evidently intersects the curve in only one point; therefore, as before shown, lines parallel to either asymptote cut the curve in a single point. By combining either of these equations with the equations of two conjugate hyperbolas, it will be found that *a line parallel to an asymptote cuts conjugate hyperbolas in points equally distant from the other asymptote.*

The ordinate of the middle point of a chord is $\frac{1}{2}b$; that is, half the intercept on the axis of Y. Hence this point is equidistant from the points in which the secant line cuts the axes, as well as equidistant from the points in which it cuts the curve. It follows, that *the portions of the secant, intercepted between the curve and the asymptotes, are equal.*

282. The equations $xy = \pm \frac{1}{4}c^2$ are central equations. Therefore, putting $x - x'$ and $y - y'$ in place of x and y, we find

$$xy - y'x - x'y + x'y' \mp \tfrac{1}{4}c^2 = 0,$$

for the equation of an hyperbola with centre at P′ and asymptotes parallel to the co-ordinate axes. Hence,

$$Bxy + Dx + Ey + F = 0$$

is the general equation of the hyperbola referred to axes parallel to its asymptotes.

Comparing these equations, we find

$$x' = -\frac{E}{B}, \qquad y' = -\frac{D}{B}, \qquad \pm \tfrac{1}{4}c^2 = \frac{DE - FB}{B^2}.$$

By these values we may determine the centre and distance of the foci, when the equation is given in its general form. The asymptotes and axes of the curve may now be drawn, and their inclinations together with the value of c determine the semi-axes.

If DE — FB is *positive*, the transverse axis is parallel to the straight line $x = y$, which bisects the angle between the *positive* directions of the co-ordinate axes. If it is negative, the transverse axis is parallel to $x = -y$. The condition that the equation shall represent straight lines is

$$DE - FB = 0.$$

EXAMPLES.—Determine the equation by the conditions that the curve shall pass through (1, 2), (2, 9) and (— 1, 0); and give the equation of its transverse axis.

What is represented by $2xy - 4x + 3y - 6 = 0$?

283. The equation of the tangent at P_1 to the hyperbola $xy = \pm \frac{1}{4}c^2$, is $xy_1 + yx_1 = \pm \frac{1}{2}c^2$. Hence, for the hyperbola whose centre is at P', we have, by substituting $x - x'$ for x, $x_1 - x'$ for x_1, etc.,

$$xy_1 + yx_1 - y'(x + x_1) - x'(y + y_1) + 2(x'y' \mp \tfrac{1}{4}c^2) = 0.$$

Then, substituting for $-y'$, $-x'$ and $x'y' \mp \frac{1}{4}c^2$ their general values $\frac{D}{B}$, $\frac{E}{B}$ and $\frac{F}{B}$, found by comparing the equations of the last Article, we have

$$\tfrac{1}{2}B(xy_1 + yx_1) + \tfrac{1}{2}D(x + x_1) + \tfrac{1}{2}E(y + y_1) + F = 0.$$

This is a formula for the tangent to $Bxy + Dx + Ey + F = 0$ at the point P_1 on the curve. If P_2 satisfies this equation, it is connected with P_1 by a *reciprocal* relation, so that P_1 and P_2 may be defined as points *reciprocally polar*. But it is shown in Art. 277, that this reciprocal property, and the fact that the polar of a point on the curve is the tangent at that point, furnish geometrical constructions of the polar of a point which are independent of the position of the co-ordinate axes. Therefore the equation is in general that of the polar of P_1, with respect to the hyperbola $Bxy + Dx + Ey + F = 0$.

CHAPTER VIII.

GENERAL EQUATION OF THE SECOND DEGREE.

284. THE most general equation of the second degree, between the co-ordinates x and y, is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

in which the coefficients and absolute term are constants which may have any values. It is the object of this Chapter to investigate, in a general manner, the locus or line represented by an equation of this form.

For this purpose we shall use the method of "arbitrary transformation" referred to in Art. 98; that is, we shall use the formulæ of transformation, regarding the constants introduced by them into the equation as arbitrary quantities to be determined in such a manner as to simplify the equation as much as possible. The transformations used are of two kinds; the first producing a change in the position of the origin, the second a change in the direction of the axes. The arbitrary quantities are the co-ordinates of the new origin, and the inclinations of the new axes. If they can be so determined as to make certain terms disappear from the transformed equation, it will take forms with which we are already familiar. Thus, if the term containing xy can be made to disappear, the equation will be known to represent a conic section; if the terms containing x^2 and y^2 can be made to disappear at once, it must represent an hyperbola, whose asymptotes are parallel to the new axes.

285. The general equation includes the equations of all conic sections; for, as noticed in Art. 265, the term containing xy may be introduced by transformation to axes having new directions. We shall therefore, first establish a criterion by which we may distinguish between the equations of the ellipse, parabola and hyper-

bola, when given in the general form. This criterion will be furnished by the equation which gives the intersection of the curve by a straight line. For we have seen that there are two directions in which if a straight line be drawn it will cut the hyperbola in a single point, while for the parabola these directions coincide, and for the ellipse they do not exist. As in Art. 237, these directions will be found by the condition that the equation by which the intersection is found shall reduce to the first degree.

Now if we substitute $y = mx + b$ in the general equation, we shall have a quadratic equation for x, in which the terms of the second degree are

$$(A + Bm + Cm^2)x^2.$$

Therefore the condition that a straight line shall cut the curve in a single point, is $A + Bm + Cm^2 = 0$,* or solving the equation,

$$m = \frac{-B \pm \sqrt{B^2 - 4AC}}{2C}.$$

If the quantity under the radical sign is positive, there are two values of m satisfying the condition, if it is zero there is but one, and if it is negative there are none. Hence,

for the hyperbola, $B^2 - 4AC > 0$;
for the parabola, $B^2 - 4AC = 0$;
for the ellipse, $B^2 - 4AC < 0$.

The equation of the hyperbola, in Art. 282, satisfies the above condition, because B^2 is essentially positive, and A and C, in that case, are each equal to zero. When $B = 0$, we find, as in the last Chapter, that for the hyperbola A and C must be of opposite signs; for the ellipse they must be of the same sign, and for the parabola one of them must be zero.

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* When the coefficient of the highest term in an equation vanishes, so that its degree is reduced, in a special case, the root which disappears becomes infinite. Thus, given the equation $ax^2 + bx + c = 0$; put $x = \frac{1}{z}$, and we have $a + bz + cz^2 = 0$ to determine z. When a vanishes, one value of z is zero, therefore one value of x is infinite. This is, therefore, the general method of ascertaining whether a curve has infinite branches.

286. The equation of a straight line passing through the origin is of the form $y = mx$. When the values of m, found in the last Article, are possible, there are two lines of this form which cut the curve in a single point. If we put $\frac{y}{x}$ for m in the equation $A + Bm + Cm^2 = 0$ from which the values of m were derived, we shall have an equation satisfied by any point on either of these lines. Making the substitution and clearing of fractions, we find

$$Ax^2 + Bxy + Cy^2 = 0,$$

which is therefore the compound equation of the two lines. In other words, if we place the terms of the second degree equal to zero, we shall have the equation of a pair of straight lines, each of which cuts the curve in a single point. Their direction ratios are the values of m in the last Article. They will be real straight lines for any equation representing an hyperbola, but imaginary for one representing an ellipse.

In an equation representing a parabola the terms of the second degree will form a complete square. For if $B^2 = 4AC$, we have $m = -\frac{B}{2C} = -\frac{2A}{B}$, and the expression $C(y - mx)^2$ when expanded may be reduced to $Ax^2 + Bxy + Cy^2$. In this case $Ax^2 + Bxy + Cy^2 = 0$ will represent two *coincident* lines of the form $y = mx$. When $B = 0$ and $A = 0$, this equation becomes $Cy^2 = 0$ or $y^2 = 0$, representing lines coincident with the axis of X; when $B = 0$ and $C = 0$, it becomes $Ax^2 = 0$ or $x^2 = 0$, representing lines coincident with the axis of Y.

Examples.—Determine the straight lines represented by each of the following equations: $x^2 - 3xy + 2y^2 = 0$, $4x^2 + 4xy + y^2 = 0$, $2x^2 - xy = 0$, $x^2 - xy + y^2 = 0$, $m(x^2 - y^2) + (m^2 - 1)xy = 0$.

Change of Origin.

287. We shall now proceed to apply the formulæ of transformation, and first, those of Art. 85, for passing to a new origin, which are

$$x = X + x' \qquad \text{and} \qquad y = Y + y',$$

in which x' and y' are the co-ordinates of the new origin.

Substituting these values in the general equation, and denoting the coefficients of corresponding terms by the same letters, we shall find that the values of A, B and C are unchanged, and that the new values of D, E and F are

$$D' = 2Ax' + By' + D, \qquad E' = 2Cy' + Bx' + E$$

and $$F' = Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F.$$

Hence, transformation to a new origin does not affect the terms of the second degree, but may be used to make two of the other terms vanish. It is evident that F' will be zero when the new origin satisfies the given equation. But the most symmetrical, as well as the simplest result, is obtained by making $D' = 0$ and $E' = 0$, which gives two equations of the first degree, for x' and y',

$$2Ax' + By' + D = 0 \qquad \text{and} \qquad 2Cy' + Bx' + E = 0.$$

Solving these equations we find

$$x' = \frac{2CD - BE}{B^2 - 4AC} \qquad \text{and} \qquad y' = \frac{2AE - BD}{B^2 - 4AC}$$

for the co-ordinates of the point to which the origin must be transferred, in order that the terms of the first degree may vanish from the equation.

EXAMPLES.—To what origin must we transform $2x^2 + 3xy + y^2 + 2x - 5y + 6 = 0$, in order to make the first powers of x and y disappear? Find the corresponding value of F', and verify by actual transformation.

288. If $B^2 - 4AC = 0$, these values of x' and y' are infinite; in that case, therefore, it is impossible to reduce the equation to the form $Ax^2 + Bxy + Cy^2 + F' = 0$. If $B = 0$, the values of x' and y' reduce to those of the co-ordinates of the centre, in Art. 263, and if both $A = 0$ and $C = 0$, they reduce to those of Art. 282.

By means of the values of D', E' and F', we may compute the coefficients of the equation for any new origin directly by substitution of the given values of x' and y'. We may use the calculated values of D' and E' to assist in finding that of F'; for $D'x' + E'y' = 2Ax'^2 + 2Bx'y' + 2Cy'^2 + Dx' + Ey'$, to which it is only necessary to add $Dx' + Ey' + 2F$ to make it the value of $2F'$. Hence

$$F' = \tfrac{1}{2}(D' + D)\,x' + \tfrac{1}{2}(E' + E)y' + F.$$

This expression for F' may be used after we have found the values of D' and E'.

Examples.—What does the equation $3x^2 - 2xy + y^2 - 5x + 2y - 9 = 0$ become, when the origin is transferred to $(2, -1)$? to $(0, 6)$? to $(3, 3)$?

What when reduced to the form $Ax^2 + Bxy + Cy^2 + F' = 0$?

289. The value of F' in the reduced form

$$Ax^2 + Bxy + Cy^2 + F' = 0$$

is readily computed by the formula in the last Article, because D' and E' are, in this case, each equal zero. To express F' in terms of the coefficients, substitute for x' and y' the values found in Art. 287. The result is

$$F' = \frac{\tfrac{1}{2}D\,(2CD - BE) + \tfrac{1}{2}E\,(2AE - BD)}{B^2 - 4AC} + F$$

$$= \frac{AE^2 + CD^2 + FB^2 - BDE - 4ACF}{B^2 - 4AC}.$$

When the numerator of this value is zero, the transformed equation takes the form $Ax^2 + Bxy + Cy^2 = 0$, which we found, in Art. 286, to represent two real or imaginary straight lines. Therefore

$$AE^2 + CD^2 + FB^2 - BDE - 4ACF = 0$$

is the condition which must be fulfilled by the coefficients when the equation represents straight lines. If $B = 0$, this condition reduces to that of Art. 261; if $A = 0$ and $C = 0$, it reduces to that of Art. 282.

Examples.—What value must be given to A, in order to make $Ax^2 + xy + 2y^2 + x - 5y + 2 = 0$ represent a pair of straight lines? (Substitute the values of the given coefficients in the condition and determine that of A.)

Determine the values of B, for which $x^2 + Bxy + 2y^2 + y - 1 = 0$ represents straight lines; and reduce each of the resulting equations to the form $Ax^2 + Bxy + Cy^2 = 0$ by transformation.

Form the compound equation of the straight lines $2x = 3y - 1$ and $x + y = 2$ (see Art. 81), and show that it satisfies the above condition; also that the point (x', y') determined by the formulæ of Art. 287, is the intersection of the given lines.

Change in Direction of Axes.

290. We shall apply the formulæ for a change in the direction of the axes to the terms of the second degree only, in the general equation, for we have seen that these terms are not affected by a change in the position of the origin. The formulæ (Art. 90) are

$$x = \frac{X \sin(\omega - \alpha) + Y \sin(\omega - \beta)}{\sin \omega}$$

$$y = \frac{X \sin \alpha + Y \sin \beta}{\sin \omega},$$

in which ω denotes the angle between the *old* axes, and is therefore *fixed*, while α and β denote the inclinations of the new axes to the old axis of X, and are therefore arbitrary.

Before making the general application of these formulæ, we shall consider the case in which the axis of Y only is changed. The value of α will then be zero, and the formulæ reduce to

$$x = X + \frac{Y \sin(\omega - \beta)}{\sin \omega}, \qquad y = Y \frac{\sin \beta}{\sin \omega}.$$

Substituting these values for x and y, in $Ax^2 + Bxy + Cy^2$, and denoting the new coefficients by A′, B′ and C′, we find

$$A' = A, \qquad B' = \frac{2A \sin(\omega - \beta) + B \sin \beta}{\sin \omega},$$

$$C' = \frac{A \sin^2(\omega - \beta) + B \sin(\omega - \beta) \sin \beta + C \sin^2 \beta}{\sin^2 \omega}.$$

The coefficient of x^2 is therefore unchanged by this transformation.

291. To make $B' = 0$, we must give to β such a value that $2A \sin(\omega - \beta) + B \sin \beta = 0$, or $\dfrac{\sin \beta}{\sin(\omega - \beta)} = -\dfrac{2A}{B}$. Now $\dfrac{\sin \beta}{\sin(\omega - \beta)}$ expresses the direction ratio of the new axis of Y. For let $y = mx + b$ be the equation, as referred to the old axes, of a

line parallel to the new axis of Y; then we know that the transformed equation of this line will be of the form $X = a$. Making the transformation, we have

$$\frac{\sin \beta - m \sin (\omega - \beta)}{\sin \omega} Y - mX - b = 0,$$

in which, therefore, the coefficient of Y must equal zero. Hence $\frac{\sin \beta}{\sin (\omega - \beta)} = m$, the old direction ratio of the line, or of the new axis of Y, which is parallel to it. Therefore we shall have $B' = 0$, or the term containing xy will vanish from the equation of the second degree, if the axis of Y be made parallel to a line whose direction ratio is $-\frac{2A}{B}$, the axis of X being unchanged.

The above expression for the direction ratio of a line, in terms of its inclination to the axis of X, will be frequently used in the following Articles. The direction ratio of the new axis of X, whose inclination is a in the more general formulæ, is $\frac{\sin a}{\sin (\omega - a)}$.

292. Since the term containing xy may thus always be made to vanish by transformation, it appears that every equation of the second degree must represent a conic section. The value of the quantity $B^2 - 4AC$ determines to which of the three classes of conics the locus of a given equation belongs; since, by Art. 285, it is positive for an hyperbola, zero for a parabola and negative for an ellipse. To show that the sign of this quantity is unchanged by transformation, we find the value of $B'^2 - 4A'C'$. Substituting the values of A', B' and C', and reducing, we find $\frac{B^2 \sin^2 \beta - 4AC \sin^2 \beta}{\sin^2 \omega}$ for the value of this expression. Therefore

$$\frac{B'^2 - 4A'C'}{\sin^2 \beta} = \frac{B^2 - 4AC}{\sin^2 \omega}.$$

Since the direction of the axis of X has not been changed by this transformation, β, which denotes the inclination of the new axis of Y, is the angle between the new axes, while ω is the angle between the old. Therefore the value of the quantity $\frac{B^2 - 4AC}{\sin^2 \omega}$

is not affected by a change in the direction of the axis of Y. In the same manner, it may be shown to remain unaltered when the axis of X is changed. Now we may transform an equation to new axes passing through the same origin, by making the required change, first in the axis of Y, and then in that of X. The result will be the same as if we had made the transformation at once by the general formulæ; for the equations found by the two methods will represent the same line, and will contain the same absolute term, and therefore must be identical. Therefore in general, *the value of the quantity* $\dfrac{B^2 - 4AC}{\sin^2 \omega}$ *is unchanged by transformation of co-ordinates.*

Since the denominator of this quantity is essentially positive, the numerator can never change sign.

EXAMPLES.—Determine the class to which each of the following conics belongs: $2x^2 - xy - y^2 + 6x + y - 5 = 0$, $3xy - y^2 + 7x = 0$, $2x^2 - 4xy + 2y^2 - x = 0$.

Transform the first, from axes making an angle of 60°, to axes bisecting their angles, and find the value of $\dfrac{B^2 - 4AC}{\sin^2 \omega}$ both before and after the transformation. (The formulæ for this transformation are given at the end of Art. 90.)

293. We now substitute the general values of x and y in the expression $Ax^2 + Bxy + Cy^2$, and, denoting the new co-efficients by A′, B′ and C′, we find

$$A' = \frac{A \sin^2 (\omega - \alpha) + B \sin (\omega - \alpha) \sin \alpha + C \sin^2 \alpha}{\sin^2 \omega},$$

$$B' = \frac{2A \sin (\omega - \alpha) \sin (\omega - \beta) + B \sin (\omega - \alpha) \sin \beta + B \sin (\omega - \beta) \sin \alpha + 2C \sin \alpha \sin \beta}{\sin^2 \omega},$$

$$C' = \frac{A \sin^2 (\omega - \beta) + B \sin (\omega - \beta) \sin \beta + C \sin^2 \beta}{\sin^2 \omega}.$$

The value of A′ involves the angle α only, and that of C′ involves the angle β only. Therefore the coefficient of x^2, for a given conic, is independent of the direction of the axis of Y, and that of y^2 is independent of the direction of the axis of X.

294. To make $A' = 0$, we must have $A \sin^2 (\omega - \alpha) + B \sin (\omega - \alpha) \sin \alpha + C \sin^2 \alpha = 0$, or, dividing by $\sin^2 (\omega - \alpha)$

and denoting $\frac{\sin \alpha}{\sin (\omega - \alpha)}$, which is the direction ratio of the new axis of X, by m,

$$A + Bm + Cm^2 = 0.$$

This is the same quadratic for m which we found, in Art. 285, for the direction ratios of straight lines which cut the curve in a single point. Since the resulting values of m are imaginary when $B^2 - 4AC$ is negative, it is impossible to make the term containing x^2 vanish from the equation of an ellipse. For the hyperbola, this term will vanish *when the axis of* X *is parallel to either asymptote*, and for the parabola, it will vanish when this axis is parallel to the axis of the curve.

The condition $C' = 0$ gives the same equation to determine the direction ratio of the axis of Y. Since there are *two* values of m which satisfy the equation when $B^2 - 4AC$ is positive, both x^2 and y^2 can be made to disappear from the equation of an hyperbola. In the case of the parabola, either one of these terms can be made to vanish; but only one, because the condition is satisfied by but one value of m, and the co-ordinate axes cannot have the same direction.

EXAMPLES.—Determine, for each of the conics in the examples under Art. 292, the direction ratios of the new axes, for which x^2 and y^2 will disappear from the equations.

295. Finally, to make the term containing xy disappear (and thus reduce the equation to the form discussed in the last Chapter), we must have $B' = 0$, or

$$2A \sin (\omega - \alpha) \sin (\omega - \beta) + B \sin (\omega - \alpha) \sin \beta + B \sin (\omega - \beta) \sin \alpha + 2C \sin \alpha \sin \beta = 0.$$

This equation, therefore, expresses a condition which is fulfilled by α and β, when the new axes are parallel to a pair of conjugate diameters. It is, then, the general relation between the inclinations of conjugate diameters. To express it in the form of a relation between direction ratios, divide by $\sin (\omega - \alpha) \sin (\omega - \beta)$, and denote $\frac{\sin \alpha}{\sin (\omega - \alpha)}$ and $\frac{\sin \beta}{\sin (\omega - \beta)}$ by m and m'. The result is

$$2A + B(m + m') + 2Cmm' = 0.$$

This is the general relation between the direction ratios of conjugate diameters as referred to any co-ordinate axes. Making $B = 0$, this equation expresses that the product of these direction ratios is constant, when the co-ordinate axes are parallel to a pair of conjugate diameters. Making $A = 0$ and $C = 0$, it expresses that the direction ratios are the negatives one of the other, when the axes are parallel to the asymptotes. Compare Arts. 205 and 279.

The equation also shows that each asymptote is conjugate to itself; for, making $m' = m$, we have the equation $A + Bm + Cm^2 = 0$ to determine the direction ratio of a line conjugate to itself. But this is the equation whose roots are the direction ratios of the asymptotes.

EXAMPLES.—Find the relation between the direction ratios of the conjugate diameters of $2x^2 - 3xy + 5y^2 + 2x - y + 4 = 0$, of $x^2 - xy + y^2 - x + y = 0$.

THE CENTRAL EQUATION.

296. We are now prepared to examine the forms to which it is possible to reduce the equation of any conic section by transformation of co-ordinates, and the relations between the curve and the co-ordinate axes, which are implied by the form of the equation.

In Art. 287, we found that by change of origin it is generally possible to reduce a given equation to the form

$$Ax^2 + Bxy + Cy^2 + F' = 0.$$

The new origin is the centre of the curve; for the term Bxy can now be made to vanish by a change in the direction of the axes, and then the equation will take the form of the central equation either of the ellipse or of the hyperbola. The general co-ordinates of the centre were found, in Art. 287, by means of two equations, which express that it is on each of the lines

$$2Ax + By + D = 0 \qquad \text{and} \qquad 2Cy + Bx + E = 0.$$

Therefore these lines are generally diameters, whose intersection is the centre.

297. In the case of the parabola these lines are parallel, but they are still diameters; for their direction ratios are $-\dfrac{2A}{B}$ and $-\dfrac{B}{2C}$;

but when $B^2 - 4AC = 0$, these quantities are equal, and by Art. 286, each of them expresses the direction ratio of the lines which cut the parabola in a single point. By the principle of combined equations,

$$2Ax + By + D + k(2Cy + Bx + E) = 0$$

will represent a series of lines passing through the intersection of these diameters in the cases of the ellipse and hyperbola, or parallel to them in that of the parabola. Hence it is the general equation of a diameter to the given conic.

The ellipse and the hyperbola, for which the diameters intersect, are sometimes called *central curves*, in distinction from the parabola, for which the diameters are parallel. Since the co-ordinates of the centre become infinite for the parabola, it is generally impossible to reduce a given equation to the central form, if $B^2 - 4AC = 0$.

298. The reduction, however, may be made, if the coefficients of the given equation fulfil the condition $2CD - BE = 0$, as well as $B^2 - 4AC = 0$. For these two conditions will make the equations $2Ax + By + D = 0$ and $2Cy + Bx + E = 0$ represent *the same line*. In this case all the diameters will be *coincident*, and the general co-ordinates of the centre take the indeterminate form. The centre may therefore be taken anywhere upon the single diameter. The numerator of the expression for F', in Art. 289, in which general co-ordinates of the centre were used, now becomes zero. Therefore the condition which generally makes $F' = 0$, and shows that the equation represents straight lines, is fulfilled in this case, but the expression for F' is indeterminate. Its value may, however, be found by substituting the co-ordinates of the assumed centre in the expression in Art. 287, and thus the equation will be reduced to the central form, $Ax^2 + Bxy + Cy^2 + F' = 0$.

299. When the central equation fulfils the condition of the parabola, $B^2 - 4AC = 0$, it may be written in the form

$$C(y - mx)^2 + F' = 0,$$

in which $m = -\dfrac{B}{2C} = -\dfrac{2A}{B}$. Compare Art. 286. This is equivalent to $y = mx \pm \sqrt{-\dfrac{F'}{C}}$, and therefore represents two parallel lines equally distant from $y = mx$. Therefore the equation of a true parabola cannot be expressed in the central form. But the equation of *a pair of parallel lines* fulfils the general condition of the parabola; and such lines constitute a conic, all of whose diameters coincide with the line midway between the parallels, and whose centre is any point of that line.

If F' and C are of the same sign, the parallel lines are *imaginary*, because the radical in the value of y is then imaginary.

300. The intercepts of $Ax^2 + Bxy + Cy^2 + F' = 0$ are $x_0 = \pm\sqrt{-\frac{F'}{A}}$, $y_0 = \pm\sqrt{-\frac{F'}{C}}$. If $F' = 0$, the intercepts are all zero, and the equation represents two real coincident or imaginary lines, as shown in Art. 286. Two intersecting straight lines, therefore, constitute a conic of which the point of intersection is the centre. The imaginary straight lines may be regarded as having a real point of intersection at the centre, which is in fact the only real point which satisfies the equation. Their separate equations are of the form $y = mx$, having imaginary values of m; but the equations of the imaginary parallel lines referred to in the last Article are of the form $y = mx + b$, having real and equal values of m, but imaginary values of b.

Since for the *ellipse* $B^2 - 4AC$ is negative, while B^2 is essentially positive, A *and* C *must have the same sign.* If F' has the opposite sign, the intercepts are both real and the ellipse is real; but if F' has the same sign, the intercepts are imaginary, and the equation has no locus; that is, it is satisfied by no real points.

If $B^2 - 4AC > 0$, A and C may have opposite signs; therefore, for the hyperbola, either or both pairs of intercepts may be imaginary; and since A or C may be zero, either or both may become infinite.

EXAMPLES.—Reduce to the central form and discuss the equations, $2x^2 - 8xy + 8y^2 + 3x - 6y = 0$, $2x^2 - xy + y^2 + 3x - 5y + 4 = 0$, $9x^2 + 6xy + y^2 + 12x + 4y + 5 = 0$, $x^2 + xy - y^2 + 3x = 0$.

301. The central equation may be further simplified by the proper changes in the direction of the axes. Thus, in Art. 291, we found that, by making the axis of Y parallel to a line whose direction ratio is $-\frac{2A}{B}$, the term containing xy will be made to

vanish. The diameter $2Ax + By + D = 0$ (see Art. 296) has this direction ratio. Therefore if the equation of a conic be referred to this diameter as axis of Y, and a diameter parallel to the axis of X, it will take the form

$$A'x^2 + C'y^2 + F' = 0.$$

Hence $2Ax + By + D = 0$ *is the diameter conjugate to that parallel to the axis of* X. In like manner, it might be shown that $2Cy + Bx + E = 0$ *is the diameter conjugate to that parallel to the axis of* Y. But, by Art. 295, $B' = 0$ whenever the axes are made to coincide with lines whose direction ratios are connected by the relation $2A + B(m + m') + 2Cmm' = 0$. Therefore in general, the equation of a conic will take the above form, when referred to the lines

$$y - y' = m(x - x') \qquad \text{and} \qquad y - y' = m'(x - x'),$$

in which x' and y' are the co-ordinates of the centre, if m and m' satisfy this relation; that is, if

$$m' = -\frac{2A + Bm}{B + 2Cm};$$

hence, when this condition is fulfilled, the lines are conjugate diameters. Since the value of m may be assumed at pleasure, only three conditions are thus imposed on the new axes, and only three of the terms of the general equation have been made to disappear.

If $m = -\frac{2A}{B}$, we find $m' = 0$, and if $m = -\frac{B}{2C}$, we find $m' = \infty$. These results agree with what is shown above respecting the diameters of Art. 296.

302. In Art. 294, it was shown that the term containing x^2 will disappear when the axis of X is parallel to an asymptote. In a similar manner the term containing y^2 can be made to disappear. Therefore if a conic be referred to its centre and an asymptote, its equation will take the form

$$B'xy + C'y^2 + F' = 0, \qquad \text{or} \qquad A'x^2 + B'xy + F' = 0,$$

according as the asymptote is made the axis of X or that of Y.

The equation $y - y' = m(x - x')$ represents an asymptote when m is one of the roots of $A + Bm + Cm^2 = 0$.

The term $B'xy$ cannot be made to disappear when one axis is an asymptote; because, as shown in Art 295, if m has one of these values, we shall find $m' = m$; but the co-ordinate axes must intersect, and therefore must have different direction ratios.

If the two asymptotes are taken as axes, the equation will take the form

$$B'xy + F' = 0,$$

and this is the only way in which four terms of the general equation can be made to disappear.

The equations of hyperbolas only can be expressed in the forms of this Article. If $F' = 0$, the first equation may be written in the form $(B'x + C'y)y = 0$, which represents the two straight lines $B'x + C'y = 0$ and $y = 0$, the latter being the axis of X. Under the same supposition, the second equation becomes $x(A'x + B'y) = 0$, which represents two straight lines, of which one is $x = 0$, the axis of Y; and the third equation becomes $xy = 0$, which represents both axes.

The Conic Referred to a Tangent.

303. Another method of simplifying the equation of a given conic, by transformation, is to make the absolute term and one of the terms of the first degree disappear. It is not always possible to do this by a change of origin only; for, if we put the values of D' and F' (Art. 287), or those of E' and F', equal to zero, we shall have two equations which, since one of them is of the second degree, may give imaginary values of x' and y'. However, the absolute term will disappear, if the new origin be taken at any point of the curve; and we shall find that one of the terms of the first degree may be made to vanish by giving a proper direction to one of the axes.

Let P_1 denote a point on the given conic; then, denoting the new values of D and E by D_1 and E_1, the equation referred to P_1 as origin is

$$Ax^2 + Bxy + Cy^2 + D_1x + E_1y = 0;$$

for, referring to Art. 287, the new value of F is zero, because P_1 satisfies the given equation. By the same Article we have

$$D_1 = 2Ax_1 + By_1 + D \quad \text{and} \quad E_1 = 2Cy_1 + Bx_1 + E.$$

It is evident that the equation of an imaginary ellipse, which is satisfied by no real points, cannot be put in the above form, which represents only real loci passing through the origin.

If $D_1 = 0$, the new origin is on the diameter $2Ax + By + D = 0$, because its co-ordinates satisfy this equation. In like manner, if $E_1 = 0$, it is on the diameter $2Cy + Bx + E = 0$.

304. We may now make the term containing y disappear, by changing the direction of the axis of Y. Using the simpler formulæ of Art. 290, the terms of the second degree receive the new coefficients of that Article, but do not give rise to new terms of the first degree. We have now to examine the effect of this transformation on the terms of the first degree.

In Art. 291, we saw that the coefficient of y will vanish from the equation of a straight line, $y = mx + b$, if the axis of Y be made parallel to it; that is, coincident with $y = mx$. Now $D_1x + E_1y = 0$ is the equation of a straight line passing through the origin. Therefore, if the axis of Y be turned until it coincide with this line, the terms of the first degree, $D_1x + E_1y$, will reduce to D_1x, and the equation will take the form

$$Ax^2 + B'xy + C'y^2 + D_1x = 0.$$

The coefficient of x is unchanged by this transformation, as it is in the example of Art. 291.

In an equation of this form, *the axis of* Y *is a tangent to the curve at the origin.* For, making $x = 0$, we find $y_0{}^2 = 0$, or both values of the intercept on the axis of Y become zero, therefore this axis meets the curve at the origin, in *two coincident points.*

305. We see then that the straight line

$$D_1x + E_1y = 0$$

is tangent to $Ax^2 + Bxy + Cy^2 + D_1x + E_1y = 0$, at the origin. That is, *if, in a given equation containing no absolute term, we put the terms of the first degree equal to zero, the result will be the equation of a tangent to the curve at the origin.* Accordingly, for an equation of the form given in the last Article, the tangent is the line $D_1x = 0$ or $x = 0$, the axis of Y. If the axis of X had been made to coincide with the tangent, the equation would have taken the form $A'x^2 + B'xy + Cy^2 + E_1y = 0$, and $y = 0$ would have been the new equation of the tangent.

It is impossible to make both of the terms of the first degree disappear in this manner, for that would require the two axes to coincide. But in an equation of the form $Ax^2 + Bxy + Cy^2 = 0$, which we have seen represents two real coincident or imaginary straight lines passing through the origin, both axes may be considered as tangents. In fact, in this case, every line passing through the centre or intersection of the straight lines fulfils the algebraic condition of tangency, namely, that of meeting the conic in two coincident points.

In the particular case when the conic becomes two *coincident* lines, every straight line which meets it is a tangent.

Examples.—Give the equations of tangents to $3x^2 - xy + y^2 + 5x - y = 0$, $2x^2 - 4xy + 2y^2 - 3x = 0$, $x^2 - 4xy + 4y^2 = 0$, $x^2 + xy = x$, $2y^2 = 0$, $(y - 5x + 2)^2 = 0$.

306. If we now change the direction of the axis of X in $Ax^2 + B'xy + C'y^2 + D_1x = 0$, we may write for the new equation

$$A'x^2 + B'xy + C'y^2 + D'x = 0.$$

The equation remains of the same form, because the axis of Y is still a tangent at the origin. The coefficient of x takes a new value, which we here denote by D′. The values of A′, B′ and C′ are those of Art. 293, for we proved in Art. 292 that the result of changing the direction first of one axis and then of the other is the same as that of using the general formulæ of transformation. Since the axis of Y was made to coincide with $D_1x + E_1y = 0$, the value of the angle β is fixed, and the direction ratio of the new axis of Y is

$$m = -\frac{D_1}{E_1}.$$

The value of the angle α, and consequently the direction ratio of the new axis of X, is arbitrary. We may therefore generally give it a value, m', determined by the relation between m and m', in Art. 295. The term containing xy can therefore be made to disappear, and the equation will then take the form

$$A'x^2 + C'y^2 + D'x = 0.$$

When the equation is in this form, the axis of Y is a tangent, and *the axis of* X *is a diameter;* because every value of x now gives two values of y numerically equal and with opposite signs.

307. The only case in which $Ax^2 + Bxy + Cy^2 + D_1x + E_1y = 0$ cannot be reduced to this last form is when m is one of the roots of $A + Bm + Cm^2 = 0$; for, as mentioned in Art. 295, we should then find $m' = m$. In this case, when we turn the axis of Y into coincidence with the tangent $D_1x + E_1y = 0$, the term containing y^2 will disappear (see Art. 294). Therefore the equation will take the form

$$Ax^2 + B'xy + D_1x = 0,$$

no change being made in the axis of X. This may be written in the form $x(Ax + B'y + D_1) = 0$, showing that it represents two straight lines,* one of which is $x = 0$, the axis of Y. When the axis of X is made tangent, the similar form $B'xy + Cy^2 + E_1y = 0$ may occur, showing that the given equation represents two straight lines, with one of which the axis of X has been made to coincide.

Tangents and Diameters.

308. In reducing a given equation to the form $Ax^2 + Bxy + Cy^2 + D_1x + E_1y = 0$, only one condition was imposed upon the new origin, namely, that it shall satisfy the given equation. Consequently we have only been able to make three of the terms of the general equation vanish by this method. But, since P_1 is any point of the curve, we may now express the general equation of the tangent at any point. After the transformation of Art. 303 we found the equation of the tangent at P_1 to be $D_1x + E_1y = 0$. Transforming back to the old axes, by substituting $x - x_1$ for x and $y - y_1$ for y, we have for the equation of a tangent (supposing P_1 to be a point of the curve)

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* The condition that m shall satisfy the equation $A + Bm + Cm^2 = 0$ is $A - \frac{BD_1}{E_1} + \frac{CD_1^{\,2}}{E_1^{\,2}} = 0$, or $AE_1^{\,2} - BD_1E_1 + CD_1^{\,2} = 0$. Referring to Art. 289, it will be seen that, since $F = 0$, this is the condition for which the conic becomes a pair of straight lines.

$$D_1 (x - x_1) + E_1 (y - y_1) = 0.$$

Substituting the values of D_1 and E_1, Art. 303, we find for the general equation of the tangent at a given point, P_1,

$$(2Ax_1 + By_1 + D)(x - x_1) + (2Cy_1 + Bx_1 + E)(y - y_1) = 0.$$

309. From the value of m, the direction ratio of this line, which is $m = -\frac{D_1}{E_1}$, we derive $D_1 + mE_1 = 0$, or

$$2Ax_1 + By_1 + D + m(2Cy_1 + Bx_1 + E) = 0.$$

This equation expresses that P_1 is on the straight line

$$2Ax + By + D + m(2Cy + Bx + E) = 0,$$

which, by Art. 297, is a diameter of the curve, m taking the place of the arbitrary constant k. We have therefore found the equation of the diameter whose vertex is P_1, in terms of the direction ratio of the tangent at P_1. This is the diameter with which the axis of X coincides when the equation is reduced to the form $A'x^2 + B'y^2 + D'x = 0$. Denoting its direction ratio by m', we find, by reducing it to the form $y = m'x + b$,

$$m' = -\frac{2A + Bm}{B + 2Cm},$$

the same expression which we found in Art. 301 for the direction ratio of the conjugate diameter. Therefore the diameter passing through P_1 is conjugate to that parallel to the tangent at P_1.

In applying these equations to find the tangent and the diameter passing through a given point on the curve, the values of D_1 and E_1 must first be computed and substituted in the equation of the tangent; then the value of the direction ratio of the tangent is to be given to k in the general equation of a diameter. Thus, given the conic $x^2 + 2xy - y^2 + 3x - 2y + 5 = 0$, to find the equations of the tangent and diameter at the point $(1, 3)$, (which will be found to be the point of the curve). We find $D_1 = 11$, $E_1 = -6$; hence the tangent is $11(x - 1) - 6(y - 3) = 0$, or $11x - 6y +$

$7 = 0$. Then, since $m = \frac{11}{6}$, the diameter is $2x + 2y + 3 + \frac{11}{6}(-2y + 2x - 2) = 0$, or $17x - 5y - 2 = 0$. These results may be verified by showing that the given point satisfies each of the equations, and that the values of m and m' satisfy the relation which should exist between them.

EXAMPLES.—Find a tangent and a diameter to $2x^2 - xy - 3y^2 - 2x + y - 4 = 0$ at the point $(3 - 2)$; at each of the points where the curve cuts the axis of X.

Find a tangent and diameter to $2x^2 + xy - 3y^2 + 3x + 7y - 2 = 0$ at $(-3, -1)$, and at $(-1, 1)$; to $x^2 - 2xy + y^2 - 2x + 2y - 3 = 0$ at each of the points whose abscissa is 1.

310. A similar method may be used in finding conjugate diameters, when the value of m is given. For, putting the value of m in place of k in the general equation of a diameter, we can find the equation of the conjugate diameter; then, using the value of m' determined by the equation thus found, we can find, in a similar manner, the equation of the diameter whose direction ratio is m. Thus, for the conic $2x^2 - xy - 3y^2 - 2x + y - 4 = 0$, the general equation of a diameter, by the formula of Art. 297, is

$$4x - y - 2 + k(-6y - x + 1) = 0.$$

Let it be required to find conjugate diameters, for one of which $m = 2$. First making $k = 2$, we find for the conjugate diameter $2x - 13y = 0$, from which $m' = \frac{2}{13}$; then making $k = \frac{2}{13}$, in the same formula, we find on reducing $50x - 25y - 24 = 0$, in which $m = 2$, the given direction ratio.

Since the asymptotes are diameters each of which is conjugate to itself, the same method enables us to find the equation of an asymptote, after the proper value of m is found. Thus, given the same conic to find the asymptotes. Substituting the given coefficients in the equation $A + Bm + Cm^2 = 0$, whose roots are the direction ratios of the asymptotes, we have $2 - m - 3m^2 = 0$, solving which we find $m = \frac{2}{3}$ or -1. Putting the first value of m in place of k, and reducing, we have $10x - 15y - 4 = 0$, in which $m = \frac{2}{3}$, therefore this is the equation of an asymptote. Putting $k = -1$, we find for the other asymptote, $5x + 5y - 3 = 0$.

If the given equation represents two intersecting straight lines, the equations thus found will be the separate equations of the lines.

Examples.—Find the conjugate diameters of $5x^2 - 2xy + 3y^2 + 6x - y + 12 = 0$, of which one is parallel to $y = 2x + 1$; find the diameters for which $m = 1$, $m = -1$, $m = 0$, etc.

Find the straight lines represented by $2x^2 + xy - 3y^2 + 3x + 7y - 2 = 0$, and verify by forming the compound equation.

311. The method explained in the last Article may also be used when we require the diameter which bisects chords parallel to a given line, or which passes through the point of contact of a tangent having a given direction ratio, m. If the curve is a parabola the value found for m' will be always the same. For, in that case $B^2 = 4AC$, therefore $\frac{B}{2C} = \frac{2A}{B}$. From this it is evident that whatever be the value of m, the value of m', Art. 309, is $-\frac{B}{2C}$. This is the common direction ratio of all the diameters. If it be substituted for m, the value of m' becomes indeterminate.

To find the equation of a tangent having a given direction, we must substitute the value of m for k, in the formula for a diameter. This will give the diameter passing through the point of tangency. The co-ordinates of its vertex, or intersection with the curve, must then be found and substituted in $y - y_1 = m(x - x_1)$. There will of course be but one intersection, and but one tangent, when the given curve is a parabola. If it is a real ellipse, there will always be two; and if it is an hyperbola, the intersections may be imaginary.

Examples.—Find the diameters of $4x^2 - 4xy + y^2 - 2x = 0$ which bisect chords parallel to $y = x$, $y = -x$, $y = 2x$, etc.

Find a tangent to this curve parallel to $y = 3x$.

Find tangents to $x^2 + xy + y^2 + 2x - 2y - 8 = 0$ parallel to each of the co-ordinate axes.

Rectangular Equations.

312. The shape of a conic and the directions of its axes are most easily investigated when the co-ordinate axes are rectangular. We shall, therefore, first show how to transform an equation from oblique to rectangular co-ordinates, and then apply the formulæ for change in direction of rectangular axes.

To make the axes rectangular, let $\beta = 90°$ in the transformation of Art. 290, in which the direction of the axis of X is unchanged. Giving this value to β, the new coefficients become (since $\sin(\omega - 90°) = -\cos\omega$)

$$A' = A, \qquad B' = \frac{-2A\cos\omega + B}{\sin\omega},$$

$$C' = \frac{A\cos^2\omega - B\cos\omega + C}{\sin^2\omega}$$

In these expressions, ω is the angle between the old axes, and there is no arbitrary constant of transformation; therefore any condition imposed upon the new or rectangular coefficients will give an equivalent condition in terms of the general coefficients.

313. The condition that the general equation shall represent a circle may thus be found. For, in order that the rectangular equation shall represent a circle, we must have $B' = 0$ and $C' = A'$. (See Art. 107.) The condition $B' = 0$ gives $B = 2A\cos\omega$. The condition $C' = A'$ gives $A\sin^2\omega = A\cos^2\omega - B\cos\omega + C$. Substituting for B its value derived from the first condition, this reduces to $A = C$; therefore

$$B = 2A\cos\omega \qquad \text{and} \qquad A = C$$

are the two conditions which must be fulfilled by every equation representing a circle.

When the equation is reduced to the central form, it is evident that the second condition expresses simply that the intercepts on the axes are equal. The first condition, taken by itself, expresses that the axis of X, which was unchanged by transformation, coincides with one of the axes of the conic. It is evident that these two conditions can be fulfilled at once only by the circle.

Examples.—When the inclination of the axes is 60° (whose cosine $= \frac{1}{2}$), what are the conditions for a circle?

Give the central equation of the circle when $\omega = 60°$.

Show by the above conditions that for a circle $B^2 - 4AC < 0$.

What is the inclination of the axes when $x^2 + \sqrt{3}\,xy + y^2 = R^2$ represents a circle?

314. To pass from one system of rectangular axes to another having the same origin, we use the formulæ of Art. 88,

$$x = X\cos a - Y\sin a \qquad y = Y\cos a + X\sin a,$$

in which there is but one arbitrary quantity, a, the inclination of the new axis of X to the old.

Substituting these values, we find, for the new coefficients of the terms of the second degree,

$$\begin{aligned} A' &= A\cos^2 a + B\sin a\cos a + C\sin^2 a, \\ B' &= 2(C - A)\sin a\cos a + B(\cos^2 a - \sin^2 a), \\ C' &= A\sin^2 a - B\sin a\cos a + C\cos^2 a. \end{aligned}$$

By adding the values of A' and C', we obtain $A' + C' = A + C$; that is, *the sum of the coefficients* A *and* C *is unchanged by transformation, when the axes remain rectangular.* In Art. 292 it was shown that the quantity $\dfrac{B^2 - 4AC}{\sin^2\omega}$ is unchanged by transformation. When the axes are rectangular, this expression becomes simply $B^2 - 4AC$, since $\sin 90° = 1$. Therefore $A + C$ *and* $B^2 - 4AC$ *are two functions of the coefficients, whose values are unaffected by this transformation.*

315. In order to find the direction of an axis, of the curve, we have to find that value of a which makes $B' = 0$. By the formulæ for the sine and cosine of the double angle, $\sin 2a = 2\sin a\cos a$ and $\cos 2a = \cos^2 a - \sin^2 a$. Therefore the condition $B' = 0$ gives

$$(A - C)\sin 2a = B\cos 2a.$$

This result may also be found without the aid of trigonometrical analysis, as follows: Since the semi-diameters inclined at equal angles on each side of the major or transverse axis are equal, the lines bisecting the angles between equal diameters are the axes of the curve. Suppose the equation to be central, the intercept on the axis of X is $\sqrt{-\dfrac{F'}{A}}$. Hence if $A' = A$, the intercept on the new axis of X will be equal to that on the old. Putting $A = A'$ we derive

$$(A - C)\sin^2 a = B\sin a\cos a,$$

in which a denotes the inclination of a diameter equal to that measured on the axis of X. This equation is satisfied by making $\sin \alpha = 0$ or $\alpha = 0°$, which makes the diameter coincident with the axis of X, and also by making $(A - C) \sin \alpha = B \cos \alpha$, in which, therefore, α is twice the inclination of an axis of the curve. This is the same equation for α as that given above for 2α, in which α is the inclination of the axis. Hence when α denotes the inclination of an axis, we have

$$\tan 2\alpha = \frac{B}{A - C}.$$

Since the angles 2α and $180° + 2\alpha$ have the same tangent, the two angles α and $90° + \alpha$ (of which we may suppose α to be in the first quadrant) satisfy this equation. The inclinations of both axes are thus given by the same equation. If $B = 0$, the inclinations are $0°$ and $90°$, because the axis of X already coincides with one axis. If $A = C$, $\tan 2\alpha$ becomes infinite, $2\alpha = 90°$ and $\alpha = 45°$; unless at the same time $B = 0$, which makes the curve a circle and the direction of the axes indeterminate.

EXAMPLES.—The co-ordinate axes being rectangular, give the inclinations of the axes of $3x^2 + 2xy + y^2 + 2x - y + 3 = 0$; of $x^2 + xy + y^2 = 3 - x$; of $x^2 - 2xy + y^2 = 8$; of $xy + y^2 = 0$.

316. When the co-ordinate axes are oblique, the value of $\tan 2\alpha$ may be found by substituting in the above expression the values of A', B' and C' in Art. 312; for these are the coefficients in the rectangular equation, when referred to the same axis of X. Hence

$$\tan 2\alpha = \frac{B'}{A' - C'} = \frac{(B - 2A \cos \omega) \sin \omega}{A \sin^2 \omega - A \cos^2 \omega + B \cos \omega - C}.$$

This is the general formula for the inclination of the axes of the curve when ω, the inclination of the co-ordinate axes, is known. Thus if $\omega = 60°$, so that $\cos \omega = \frac{1}{2}$, and $\sin \omega = \frac{1}{2}\sqrt{3}$, we have in general $\tan 2\alpha = \dfrac{(B - A)\sqrt{3}}{A + B - 2C}$. Therefore in the given equation $3x^2 + 2xy + 2y^2 = 36$, $\tan 2\alpha = -\sqrt{3}$, $2\alpha = 120°$ and $\alpha = 60°$; that is, one axis of the curve coincides with the axis of Y.

By making use of the formulæ for the double angle, referred to in the last Article, the expression for $\tan 2a$ may be put in the following form:

$$\tan 2a = \frac{A \sin 2\omega - B \sin \omega}{A \cos 2\omega - B \cos \omega + C}.$$

EXAMPLES.—If $\omega = 45°$, what are the inclinations of the axes of the conic $x^2 - xy + y^2 + 2x - 2y + 1 = 0$?

Show generally, that if $A = C$, the axes of the curve bisect the angles between the co-ordinate axes.

Show that the general conditions of the circle render $\tan 2a$ indeterminate.

Prove, by the last value of $\tan 2a$ and the formulæ for the double angle, that when $B = 2C \cos \omega$, the axis of Y is parallel to an axis of the curve.

317. To find the semi-axes, it is necessary to reduce the equation to the central form, $Ax^2 + Bxy + Cy^2 + F' = 0$; and then, making the axes rectangular, to reduce it to the form,

$$A'x^2 + C'y^2 + F' = 0.$$

Whether the axes are rectangular or oblique, F' is computed by the formula of Art. 289. Now supposing the axes rectangular, the values of A' and C' may be found by means of the quantities, proved in Art. 314 to be constant in value. For since $B' = 0$,

$$A' + C' = A + C \qquad \text{and} \qquad 4A'C' = 4AC - B^2.$$

We have therefore two equations by which to find the values of A' and C'. The form of these equations is such as to give values of A' and C', which may be interchanged; the reason of which is that the axis of X may be made to coincide with either axis of the curve.

The equations may be solved thus: Squaring the first and subtracting the second member from member, we have $(A' - C')^2 = (A - C)^2 + B^2$, hence $A' - C' = \pm\sqrt{(A - C)^2 + B^2}$. Since the quantity under the radical sign is essentially positive, the values of A' and C' are always possible. To determine which sign to give to $A' - C'$, observe that, by the values in Art. 314, $A' - C = 2B \sin a \cos a$; hence *if we take a in the first quadrant we must give* *to* $A' - C'$ *the sign of* B. From the values of $A' - C'$ and $A' + C'$ we readily find those of A' and C'. The squares of the semi-axes are numerically equal to the squares of the intercepts in the reduced equation; the intercepts themselves being both real for the ellipse, one real for the hyperbola, but neither real for the imaginary ellipse.

Examples.—The axes being rectangular, transform each of the following conics to its axes; $x^2 - 4xy - 2y^2 = 36$; $5x^2 + 3xy + y^2 = 1$; $6x^2 - 5xy - 6y^2 = -26$.

Determine that semi-axis of $2x^2 + 4xy + 5y^2 + 6x - 2y = 0$ which makes an acute angle with the axis of X.

318. In order to find the values of A' and C' directly from the oblique equation, let us find the value of the constant quantity $A + C$ (in the rectangular equation) in terms of the general or oblique coefficients. Adding the values of A' and C', the rectangular coefficients of Art. 312, we find

$$A' + C' = \frac{A + C - B\cos\omega}{\sin^2\omega}.$$

Now, by Art. 314, $A' + C'$ has a constant value independent of the direction of the axis of X; therefore the expression in the second member, which involves only the mutual inclination of the axes, is constant in value. By this, and Art. 292, we see that

$$\frac{A + C - B\cos\omega}{\sin^2\omega} \quad \text{and} \quad \frac{4AC - B^2}{\sin^2\omega}$$

are two functions of the coefficients and the angle between the co-ordinate axes, of which the values are unchanged by transformation. When $\omega = 90°$, these expressions reduce to those given in the last Article; and in all cases, they give us the values of $A' + C'$ and $4A'C'$ by which A' and C' may be found.

Examples.—The inclination of the axes of co-ordinates being 60°, give the equation of each of the following conics as referred to its axes; $3x^2 + 3xy + 4y^2 - 12 = 0$; $x^2 - xy - y^2 = 1$; $2x^2 + xy + 2y^2 = 2$; $x^2 - xy + y^2 = 4$.

Transform to its axes each of the following, supposing $\omega = 120°$: $x^2 - xy + y^2 = 1$; $4x^2 - 4xy + 3y^2 - 2 = 0$; $xy - y^2 = 1$; $2x^2 - xy + y^2 = 1$.

Transform to its axes $3x^2 - 2xy - 3y^2 = 6$, when $\cos\omega = \frac{1}{3}$.

Transform to its axes $Ax^2 + Bxy + Ay^2 = 1$.

$$Ans.\ \frac{A + \frac{1}{2}B}{1 + \cos\omega}x^2 + \frac{A - \frac{1}{2}B}{1 - \cos\omega}y^2 = 1.$$

The equation of the ellipse referred to conjugate diameters is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If it be transformed to any other pair of conjugate diameters it will retain the same form, but a and b will take new values. Prove that $a^2 + b^2$ and $ab \sin\omega$ are constant. (Substitute the coefficients of x^2 and y^2 for A and C in the above expressions.)

Prove, in a similar manner, that the sum of the squares of the reciprocals of perpendicular semi-diameters is constant. (Assume the rectangular equation $Ax^2 + Bxy + Cy^2 + F = 0$, and take the intercepts on the axes for the semi-diameters.)

Conic Fulfilling Given Conditions.

319. Since the general equation of the conic section contains six coefficients, whose ratios determine the position and shape of the curve, it may be regarded as containing *five* arbitrary constants, which may be so determined as to make the conic fulfil five given conditions. Thus, a conic passing through five given points may be found; for, assuming a value for the absolute term, the values of the five coefficients may be determined by five equations of condition expressing that the curve shall pass through the five given points. Since these equations are of the first degree, they have but one solution; therefore one conic, and generally but one, can be found fulfilling the conditions. If three of the given points are in the same straight line, the conic which we shall find will consist of this straight line and that which passes through the two other points. If four of the points are in one straight line, the conic will be indeterminate, because this straight line in connection with any straight line passing through the fifth point will fulfil the conditions.

320. In general a conic can be made to fulfil any five conditions. That it shall be a *parabola* constitutes one condition, because this gives the equation $B^2 - 4AC = 0$ between the coefficients. That it shall be an ellipse is only a *restriction*, because this does not give an equation between the coefficients, but only requires that 4AC shall exceed B^2. A parabola may generally be found passing through four

given points, because four equations of condition together with $B^2 = 4AC$ serve to determine the five unknown quantities, or ratios of the coefficients. But, as this last equation is of the second degree, there may be *two solutions*. In fact, it is evident that two parabolas may intersect in four points. The solution may be impossible, because the results may be imaginary quantities. This will happen if the four points are so situated that one may be enclosed in the triangle formed by joining the others, for it is evident that four points of this character cannot be found on a parabola.

To be similar and parallel to a given conic, or to have given directions for the asymptotes, constitutes two conditions, because it determines the ratios of the coefficients of the three terms of the second degree, upon which, as we have seen, depend the directions of the asymptotes, conjugate diameters and axes. The parabola having a given direction for its diameters and axis is a case of this. If the axis is to be parallel to $y = mx + b$, the equation may be assumed in the form

$$(y - mx)^2 + Dx + Ey + F = 0,$$

in which we have assumed $C = 1$ (see Art. 286), so that the remaining coefficients may be determined by three other conditions.

To be a circle also constitutes two conditions, for the conditions of Art. 313 determine the ratios of A, B and C. If we wish to find the circle passing through three given points, we may assume the equation in the form

$$x^2 + 2xy \cos \omega + y^2 + Dx + Ey + F = 0.$$

EXAMPLES.—Determine the parabola whose axis is parallel to $y = 2x$, and which passes through $(1, -1)$ $(-2, 0)$ and $(-1, 2)$.

If $\cos \omega = \frac{1}{2}$, what is the equation of the circle passing through these points?

321. A pair of straight lines constitutes a conic fulfilling one condition, for the coefficients must satisfy the equation of Art. 289. If we attempt to determine the conic by means of this equation and four equations of condition expressing that the curve shall pass through four given points, the result of elimination will be an equation of the *third degree* containing one unknown quantity. Therefore there may be *three solutions*. The interpretation of this is that

four given points may be joined by pairs in three different ways, giving three pairs of straight lines fulfilling the conditions. The three equations of the second degree may be found by compounding the equations of the straight lines, as in Art. 81.

A pair of parallel lines is a conic fulfilling two conditions; namely, the condition for straight lines and that of the parabola. A pair of coincident lines fulfils a third condition, so that it can be made to satisfy only two more conditions, or to pass through two given points.

If $S = 0$ represent the equation of a conic, and $\alpha = 0$, $\beta = 0$, the equations of straight lines, then S is a polynomial of the second degree involving x and y, while α and β are of the first degree. Then the condition for *straight lines* is equivalent to this: that *the polynomial* S *shall be the product of two factors of the first degree*, or that the equation shall be capable of taking the form $\alpha\beta = 0$. We here use the Greek letters to denote expressions of the form $Ax + By + C$, then, since $\alpha = 0$ contains two arbitrary constants and $\beta = 0$ contains two, $\alpha\beta = 0$ contains four.

The condition for *coincident lines* is that the equation shall be reducible to the form $\alpha^2 = 0$; in other words, that *the polynomial* S *shall be the square of an expression of the first degree.*

322. If $S = 0$ and $S' = 0$ are the equations of two given conics,

$$S + kS' = 0$$

is the equation of a conic passing through their points of intersection. In Art. 270, we supposed the equations combined to be of the form in which $B = 0$; that is, the term containing xy did not appear in the equations. Therefore the given conics already fulfilled one condition, and the whole system of conics which may be represented by $S + kS' = 0$ fulfilled the same condition, as explained in Art. 271. We saw, in that case, that the conics may intersect in four points; but we could not use the formula to produce the equation of the conic passing through four *given* points, because it is generally impossible to find more than one conic of that form which shall pass through four given points.* Using now

* See Note to Art. 271, in which it was shown that three points and the condition implied in the form of the equation determine the fourth point, so that $S + kS' = 0$ does not fulfil five independent conditions. Four points

the general form of the equation, this may readily be done by taking the equations of the pairs of lines mentioned in the last Article. Thus, given four points A, B, C and D, form the compound equation of the lines AB and CD, also that of the lines AC and BD. These constitute the equations of two conics passing through the four points: they may, therefore, be used in forming the general equation $S + kS' = 0$. It is now possible so to determine k, as to make the conic pass through a fifth given point.

323. This method may be adapted to other cases of conics fulfilling four conditions. Thus, to be tangent to a given line at a given point constitutes two conditions, for it implies that the conic shall there intersect the given lines in *two coincident points.* If now we require the equation of a conic touching a given line at the point A and passing through B and C, we have only to take for $S = 0$, the compound equation of the given line and BC, and for $S' = 0$ that of the pair of lines AB and AC. For it is evident that the later conic intersects the given line only at the point A, and the conic $S + kS' = 0$ cannot meet it in any other point, because it cannot meet the first conic in a point not on the second. For instance, let the conic be required to touch $2x - y + 1 = 0$ at $(1, 3)$ and to pass through $(-1, 2)$ and the origin. The first conic is $(2x - y + 1)(2x + y) = 0$ or $4x^2 - y^2 + 2x + y = 0$, and the second is $(x - 2y + 5)(3x - y) = 0$ or $3x^2 - 7xy + 2y^2 + 15x - 5y = 0$. Combining the equations, we have

$$4x^2 - y^2 + 2x + y + k(3x^2 - 7xy + 2y^2 + 15x - 5y) = 0,$$

in which k may be determined by another condition.

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generally determine the direction of a pair of conjugate diameters. For, suppose two parabolas to pass through them. Take two straight lines, of which each is parallel to the axis of one of these parabolas. The equations of the parabolas, referred to these lines as co-ordinates axes, will be, one of the form $Ax^2 + Dx + Ey + F = 0$, the other of the form $C'y^2 + D'x + E'y + F' = 0$. Denoting these equations by $S = 0$ and $S' = 0$, we see that $S + kS' = 0$ will always be of the form in which $B = 0$; therefore every conic passing through the four points has a pair of conjugate diameters parallel to the axes of the parabolas. When the lines joining the given points form a quadrilateral with a *re-entrant* angle, hyperbolas only can be drawn through them, and there are no parallel pairs of conjugate diameters.

324. For the equation of the conic tangent to two given lines at given points, take for $S=0$ the pair of given lines, and for $S'=0$ the pair of coincident lines passing through the given points. Thus, let the conic be required to touch the axis of X at (3, 0) and the line $x=y$ at (1, 1). The first conic is $y(x-y)=0$, and the second is $(x+2y-3)^2=0$. Hence the required equation is $x^2+4xy+4y^2-6x-12y+9+kxy-ky^2=0$, in which k may be determined by another condition. For instance, required also that it be a parabola. In the equation, $A=1$, $B=4+k$, $C=4-k$, therefore $B^2=4AC$ gives $(4+k)^2=4(4-k)$, which reduces to $12k+k^2=0$. This is satisfied by $k=0$, because the equation will then reduce to that of the coincident lines which are of the form of the parabola; the other solution is $k=-12$, which gives the true parabola $x^2-8xy+16y^2-6x-12y+9=0$.

In a similar manner we can find the equation of a conic meeting a given conic in four points of which one pair or two pair may be coincident. It is also possible to make three of the points of intersection coincide; for combine with the given equation that of a pair of lines, one of which is a tangent and the other passes through the point of contact. The conic $S+kS'=0$ is in this case said to make double contact with $S=0$. By taking for $S'=0$ the square of the equation of a tangent, $S+kS'=0$ may be made to meet $S=0$ in *four* coincident points, in which case it is said to make a contact of the third order.

325. Any equation of the second degree, in which one constant is arbitrary, may be regarded as an equation of the form $S+kS'=0$, in which k represents the arbitrary constant, and S and S' are polynomials, of which at least one is of the second degree. Though $S=0$ and $S'=0$ may not actually intersect in four points, $S+kS'=0$ always fulfils four conditions. Thus, if S' is of the first degree, $S+kS'=0$ will be parallel and similar to $S=0$, and will intersect it in the two points where it is cut by the straight line $S'=0$. The equation of the parabola with axis parallel to a given line, and passing through two given points, may be formed, by taking for $S=0$ the pair of lines parallel to the given lines and passing through the given points, and for $S'=0$ the single straight line passing through the points. That of the hyperbola having asymptotes

in given directions may be found, by taking for $S = 0$ the compound equation of two straight lines, each of which passes through one of the given points and has one of the given directions.

326. The general equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ may itself be regarded as one of the conics represented by $S + kS' = 0$, when $S = 0$ is the pair of straight lines passing through the origin, $Ax^2 + Bxy + Cy^2 = 0$, Art. 286, and $S' = 0$ is the single line $Dx + Ey + F = 0$. From this we might at once infer that the lines $Ax^2 + Bxy + Cy^2 = 0$ meet the curve each in a single point, and that the terms of the first degree

$$Dx + Ey + F = 0,$$

constitute the equation of a straight line passing through the points in which these lines meet the curve. Since the pair of lines $Ax^2 + Bxy + Cy^2 = 0$ is real for the hyperbola, this line cuts the curve in the points where it is cut by two lines passing through the origin parallel to the asymptotes. For the parabola, it is a tangent to the curve at the point where a line passing through the origin parallel to the axis cuts the curve. In the case of the ellipse, it does not meet the curve.

The four conditions fulfilled by $S + kS' = 0$ may always be considered as determining four common points of intersection for the whole series. But these points may be *real, coincident or imaginary*, as explained in Art. 272; and in the special cases of similar ellipses and hyperbolas in which one or both pairs of asymptotes are parallel or coincide, some of the intersections become infinite. Compare Art. 273.

EXAMPLES.—Find the equation of the conic passing through $(3, 1)$, $(2, -2)$, $(-1, -1)$, $(0, 2)$ and $(2, -3)$ by the method of Art. 322.

Give the general equation of the conic tangent to $y = 2x - 3$ at $(2, 1)$ and passing through $(1, -2)$ and $(3, 1)$; and determine k so as to make the conic pass through the origin.

Give the general equation of the conic touching $x^2 - xy + 4y^2 + 2x - 3y + 4 = 0$, at the points where it is cut by the straight line $2x - y + 1 = 0$. What must be the inclination of the co-ordinate axes in order that a circle may be found fulfilling these conditions?

Determine the parabola parallel to $3x + 2y = 0$ and passing through $(1, 1)$, $(2, -2)$ and the origin.

If $\cos\omega = \frac{1}{3}$, what is the equation of the circle tangent to $y = x$ at the origin and passing through $(-2, 1)$?

What is represented by the general equation, supposing the coefficient A to be arbitrary?

Ans. A series of conics having common tangents at the points where they cut the axis of Y; for $x^2 = 0$ is the equation of two lines coincident with that axis. The equation of the pair of tangents may be found by determining A as in the first example under Art. 289.

Interpret the equation in a similar manner, regarding each of the coefficients in turn as the arbitrary constant.

Supposing $\alpha = 0$, $\beta = 0$, etc., to represent straight lines, what is denoted by $\alpha\beta + k\gamma\delta = 0$? by $\alpha\beta + k\gamma^2 = 0$? by $\alpha^2 + k\beta^2 = 0$? by $S + k\alpha^2 = 0$? by $\alpha\beta + k\gamma = 0$? by $\alpha\beta + k\alpha = 0$? by $\alpha^2 + k\beta = 0$? by $S + k\beta = 0$?

What is denoted by $\alpha\beta + F = 0$?

Ans. An hyperbola whose asymptotes are $\alpha = 0$ and $\beta = 0$.

If $S = 0$ and $S' = 0$ have a common asymptote, show that $S + kS' = 0$ has the same line for one asymptote. (If $\alpha = 0$ is the equation of the common asymptote, $S = 0$ and $S' = 0$ may be written in the forms $\alpha\beta + F = 0$ and $\alpha\gamma + F' = 0$; and $S + kS' = 0$ is equivalent to $\alpha(\beta + k\gamma) + F + kF' = 0$, an hyperbola whose asymptotes are $\alpha = 0$ and $\beta + k\gamma = 0$. That is, $S + kS = 0$ represents a series of hyperbolas of which $\alpha = 0$ is a common asymptote, and the other asymptotes pass through a common point, the intersection of $\beta = 0$ and $\gamma = 0$.)

EQUATION OF A POLAR.

327. The general formula for the tangent at a given point P_1 found in Art. 308, is

$$(2Ax_1 + By_1 + D)(x - x_1) + (2Cy_1 + Bx_1 + E)(y - y_1) = 0.$$

This is always the equation of a line passing through P_1, and it represents a tangent when

$$Ax_1^2 + Bx_1y_1 + Cy_1^2 + Dx_1 + Ey_1 + F = 0;$$

that is, when P_1 is a point of the curve. If we partially expand the equation of the tangent and add to it twice this equation of condition, it will take the form

$$(2Ax_1 + By_1 + D)x + (2Cy_1 + Bx_1 + E)y + Dx_1 + Ey_1 + 2F = 0.$$

Now this is *the equation of the polar of* P_1; for dividing by 2 and arranging the terms in the order of the coefficients, we have

$$Axx_1 + \tfrac{1}{2}B(xy_1 + yx_1) + Cyy_1 + \tfrac{1}{2}D(x + x_1) + \tfrac{1}{2}E(y + y_1) + F = 0.$$

When written in this form, it is evident that if P_2 is on this line, it is connected with P_1 by a *reciprocal* relation. Such points, therefore, may be defined as *reciprocally polar*, and the straight line as the *polar* of P_1. It will be seen that this is the most general formula connecting points by a reciprocal relation of the first degree with respect to the co-ordinates of each. The condition that a point shall be on its own polar is the same as that which expresses that it is on a certain conic, and the relations between the curve, pole and polar, given in Art. 277, are of general application, because they are independent of the co-ordinate axes.

328. If it be required to find the equations of tangents passing through a given point, the co-ordinates of the points of tangency may be found by combining the equations of the curve and the polar of the given point.

The polar of the origin is found by the formula to be

$$Dx + Ey + 2F = 0.$$

Comparing this with $Dx + Ey + F = 0$, which by Art. 326 passes through the points where the curve is cut by parallels to the asymptotes drawn through the origin, we see that it is parallel to that line and twice as far from the origin. As any point may be taken as origin, we derive the following property of the hyperbola: *If from a given point tangents be drawn and also lines parallel to the asymptotes, the line joining the points of section will be parallel to that joining the points of tangency and midway between it and the given point.*

CHAPTER IX.

GEOMETRICAL LOCI.

329. In this Chapter, the principles of Analytical Geometry are applied to the problem of finding the *locus*, or path, of a point moving according to a given geometrical law. The method consists in assuming co-ordinate axes, to which to refer the moving point; establishing a relation between its co-ordinates, equivalent to the law of the point's motion; and finally, interpreting the relation found, which is the equation of the locus, so as to ascertain its character and position.

The law by which the describing point moves may be stated in the form of a condition imposed upon the point, sufficient to *restrict* it to a certain line, but not to *determine* its position. Hence a locus corresponds to an "indeterminate equation," or single equation between x and y, the unknown co-ordinates of a point. See Arts. 15 and 16. We may, also, regard the law of the motion as the statement of a common property of all the points of the line sought. From this property we are to deduce the equation of the line, which also expresses a common property of all its points. If the equation found is included under any of the general equations which we have investigated—that is, if it is of the first or second degree—the form and position of the line having the given property become known. Thus, in the preliminary illustration of Art. 22, the result being the equation of a circle, we learn that it is a property of the circle that the squares of the distances of any point of the circumference from certain fixed points have a constant sum.

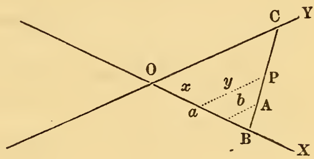
Choice of Co-ordinate Axes.

330. When the problem is simply to find the line described by a point, and its position with reference to certain fixed lines and

points, the readiness with which we can establish the equation between x and y depends upon the manner in which we assume the axes. We select as an illustration the following problem:

Given two fixed intersecting lines and a fixed point, A, *a line is drawn through* A, *meeting the fixed lines in* B *and* C; *find the locus of the middle point of* BC.

Take the two fixed lines, OX and OY, as axes, and draw the ordinate of the fixed point A, and that of the middle point P whose locus is required. Denote the co-ordinates of A, which are constant, by a and b; those of P are of course x and y. We have now to establish a relation between x, y, a and b, by means of the conditions of the problem and the geometrical principle of similar triangles. Since P is the middle point of BC, $OB = 2x$ and $OC = 2y$; and by similar triangles $2y : 2x :: b : 2x - a$; hence



$$bx = 2xy - ay.$$

This is the equation of the locus required, which is therefore an hyperbola with asymptotes parallel to the fixed lines OX and OY. The equation also shows that the locus passes through the fixed point A (a, b); and through the intersection of the fixed lines. By Art. 282, the centre of the hyperbola is the point $(\frac{1}{2}a, \frac{1}{2}b)$ midway between O and A.

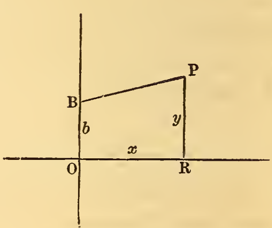
331. It is evident, on the first statement of this problem, that the conditions are *not* sufficient to fix the point P, because the line BC is not fixed. But they limit the position of P to a locus, which is described by the point as the line BC revolves about the point A. Accordingly, we find it possible to establish a *single* equation between the co-ordinates of P, which leaves its position *indeterminate*, though restricted. If the conditions had been sufficient to fix the point P, we should have been able to establish *two* equations between x and y, and their values would have been determinate.

In solving problems of this kind, it is necessary to distinguish carefully between the constant and variable lines in the figures. To do this we must consider the motion which takes place among the parts of the figure, as, in the above example, the revolution of the line BC, by reason of which OC and OB are variables. The constant distances of the figure should be denoted by letters, so that the solution may be general; the discussion of the problem then consists in examining the special cases which may occur, when the parts of the figure have particular relative positions. Thus, in the present case, if the fixed point A be on one of the fixed lines as OX, so that $b = 0$, the equation reduces to $2xy - ay = 0$, which represents the two straight lines $y = 0$ and $2x = a$. Therefore in this case the locus becomes two straight lines, one of which is parallel to OY and bisects the distance OA, and the other is the line OX itself.

If the complete revolution of BC about the point A be considered, it will be seen in what manner the two branches of the curve are described by P, in the general case. In the special case, it will be observed that the line OX is not, strictly speaking, described by the motion of P; but it constitutes part of the locus, because the position of B, which, in this case, generally coincides with A, becomes indeterminate when the revolving line coincides with OX, and P is then an indeterminate point of that line.

332. *To find the locus of a point moving in such a manner that the square of its distance from a fixed point is proportional to its distance from a fixed line.*

Take the fixed line as the axis of X, and since the condition of the problem involves distances, let the axes be rectangular; the axis of Y may be taken so as to pass through the fixed point. Then will PR, the distance from the line, be denoted by y. Let b denote the distance OB of the fixed point from the fixed line, and assume $PB^2 : PR :: c : 1$. Now $PB^2 = x^2 + (y - b)^2$, therefore $x^2 + (y - b)^2 = cy$, or



$$x^2 + y^2 - (2b + c)\,y + b^2 = 0.$$

Hence the locus of P is a circle whose centre is on the line OB, and which generally does not cut the axis of X. But in the special case when the fixed point is on the fixed line; that is, when $b = 0$, the equation becomes $x^2 + y^2 - cy = 0$, which represents a circle having for diameter c, and touching the axis of X or given line at the origin or given point.

When the conditions of a problem make two variable quantities proportional, we may always assume one of them equal to the other multiplied by a constant. Thus, in this example, $PB^2 = cPR$, in which c represents a constant third proportional to PR and PB.

EXAMPLES.—A line of fixed length moves with one of its extremities in each of two fixed lines: find the locus of any point of the line. (Let a and b be the distances of the point from the extremities of the line, and discuss the cases $a = 0$ and $a = b$.)

A line cuts two fixed lines, OX and OY, in B and C, and moves in such a manner that the area of the triangle OBC is constant; find the locus of the middle point of BC.

Find the locus of the middle point of a rectangle inscribed in a given triangle. (Assume the base of the triangle as the axis of X, and a perpendicular through the vertex as axis of Y, so that the axes shall be parallel to the sides of the rectangle.)

Given the base and sum of the sides of a triangle, a perpendicular to the base is drawn through the vertex and produced to equal one of the sides, find the locus of its extremity.

Ans. A straight line.

A point moves so that the squares of its distances from two fixed points are as $m : n$, what is the locus described?

A line is drawn through a fixed point A, cutting a fixed line in D; through the point R of the fixed line is drawn a line PR cutting AD in P; find the locus of P, supposing DR to be constant, and the line PR to cut the fixed line at a constant angle. (Take the fixed line as axis of X, and for axis of Y, a line passing through A parallel to PR.)

APPLICATION OF ANALYTICAL FORMULÆ.

333. In the foregoing examples, we have been able to establish the relation between x and y by means of simple geometrical principles. It is frequently necessary to employ the principles of analysis for this purpose, especially when the axes are already determined. For example, let it be required *to find the locus of a point, when the square of its distance from a given point* P_1 *is proportional*

to its distance from the line $x \cos a + y \sin a = p$, the axes being rectangular. Using the formulæ for the distance of points, and the distance of a point from a line, the condition of the problem gives the equation

$$(x - x_1)^2 + (y - y_1)^2 = c\,(x \cos a + y \sin a - p),$$

which when expanded evidently represents a circle. When P_1 is on the given line it may be shown that the equation represents a circle touching the given line at P_1, and having c for its diameter, as in the special case discussed in the last Article. It will be seen that the result of this method is a general formula, but that the mere solution and discussion of the problem is more simple when we are at liberty to assume the axes.

334. When a variety of points is given, the most symmetrical and useful solutions will be obtained by using general expressions for all of them. Thus: *To find the locus of* P, *when the sum of the squares of its distances from any number of fixed points is constant.*

Let $P_1, P_2 \ldots\ldots P_n$ be the points (n being the number of points), and suppose the axes rectangular; then, denoting the constant sum by c^2, we have

$$(x-x_1)^2 + (x-x_2)^2 \ldots + (x-x_n)^2 + (y-y_1)^2 \ldots + (y-y_n)^2 = c^2.$$

Expanding and dividing by n, we find this to be the equation of a circle, the co-ordinates of whose centre are arithmetical means between the corresponding co-ordinates of the given points.

Examples.—Find the locus of a point whose distance from a fixed point is proportional to its distance from a fixed line. (Take the fixed line as the axis of Y, and assume the first distance equal e times the second. The result will be a conic section of which e is the eccentricity, the fixed point the focus and the fixed line the directrix.)

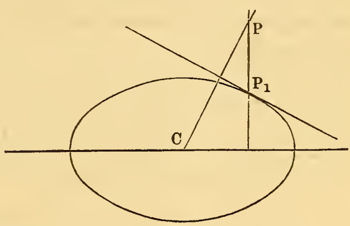
Find a formula for a conic having the point P_1 for focus, and the line $x \cos a + y \sin a = p$, for directrix.

Show that the locus of a point, the square of whose distance from one fixed line is proportional to its distance from another, is a parabola whose axis is parallel to the first line. (Assume rectangular axes and find a general formula; also assume the first line as axis of X and prove that the second line is a tangent, by Art. 326.)

Elimination of Variables.

335. In many cases it is convenient to use other variables besides x and y, and to derive from the conditions a sufficient number of equations to eliminate these auxiliary variables, so as finally to have a single equation between x and y. The methods we used in finding the equations of the ellipse and hyperbola were instances of this: *three* equations were found between the *four* variables, x, y, r and r', and were reduced to a *single* equation between *two* variables, just as in Algebra any number of equations containing an equal number of unknown quantities is reduced to a single equation containing one unknown quantity. The number of equations must be one less than the whole number of variables, in order that the position of P may be indeterminate.

As an illustration, we solve the following problem: *Find the locus of the point, in which the perpendicular from the centre of an ellipse upon a tangent cuts the ordinate of the point of contact.*



Let P be the point of intersection, and let φ be the eccentric angle of the point of contact, P_1. Then the equation of the line CP, perpendicular to the tangent (or parallel to the normal, Art. 200), and passing through the centre is

$$B \cos \varphi \,.\, y = A \sin \varphi \,.\, x.$$

Now the abscissa of P is the same as that of P_1, therefore $x = A \cos \varphi$; and since P is a point of the above line $y = \frac{A^2}{B} \sin \varphi$, hence

$$\cos \varphi = \frac{x}{A} \qquad \text{and} \qquad \sin \varphi = \frac{By}{A^2}.$$

Eliminating φ by substitution in $\sin^2 \varphi + \cos^2 \varphi = 1$, we find

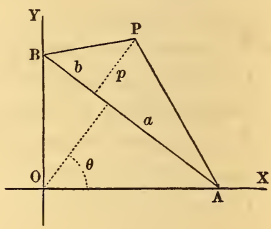
$$\frac{x^2}{A^2} + \frac{B^2 y^2}{A^4} = 1,$$

the equation of an ellipse, whose semi-axes are A and $\frac{A^2}{B}$. The latter being a third proportional to B and A, the locus is similar to the given ellipse, and its minor axis coincides with the given major axis.

336. When the motion which takes place in describing the locus involves a constant change in the direction of certain lines, it will often be necessary to use as an auxiliary variable an angle dependent upon their directions. Thus:

A given triangle moves with two of its vertices in the two rectangular axes of co-ordinates; find the locus of the third vertex.

Let PAB be the given triangle; drop a perpendicular from the origin upon the base AB, and let θ, denoting the inclination of this perpendicular, be the auxiliary variable. Since PAB is a given triangle, any of its parts may be used as constants. It will be most convenient to use the perpendicular from P, and the segments into which it divides the base, denoted by p, a and b, as in the figure. These constants and the value of θ, at any stage of the motion, determine the position of P. Dropping perpendiculars from both extremities of p, we easily show that



$$x = p\cos\theta + b\sin\theta \qquad \text{and} \qquad y = p\sin\theta + a\cos\theta.$$

These are the two relations between x, y and θ. We eliminate θ by finding values for $\sin\theta$ and $\cos\theta$ and using the relation $\sin^2\theta + \cos^2\theta = 1$. Thus, eliminating successively $\sin\theta$ and $\cos\theta$, we have

$$(p^2 - ab)\cos\theta = px - by \qquad \text{and} \qquad (p^2 - ab)\sin\theta = py - ax.$$

Squaring and adding member to member,

$$(p^2 - ab)^2 = (p^2 + a^2)x^2 + (p^2 + b^2)y^2 - 2p(a + b)xy.$$

This is the required equation between x and y: by the principles of the last Chapter, it represents an ellipse, with centre at the origin, but not having the lines OA and OB for axes.*

EXAMPLES.—Find the locus of the point in which a parallel to the axis of X drawn through P_1 (Fig. Art. 335), cuts the line PC.

Given two fixed straight lines intersecting in O, a straight line cuts them in A and B, and P is such a point of the line that $AP : BP :: m : n$. Find the locus of P, 1st, when $OA + OB$ is constant; 2d, when $OA^2 + OB^2$ is constant; 3d, when $OA \times OB$ is constant. (Let v and z, denoting respectively OA and OB, be the auxiliary variables. By similar triangles their values are readily expressed in terms of x and y.)

Find the locus of P when AB constantly passes through a fixed point. (A relation between v and z may be found by expressing the condition that the fixed point (a, b) is on the line whose intercepts are v and z.)

A straight line passing through a fixed point cuts a given conic, find the locus of the middle point of the chord. Take the fixed point as origin. Then we may assume $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ for the equation of the conic, and $y = mx$ for the straight line. The abscissas of the points of intersection are the roots of $(A + mB + m^2C)\,x^2 + (D + mE)x + F = 0$. It is not necessary to find these roots, for their sum is the negative of the coefficient of x divided by that of x^2; hence the abscissa of the middle point is

$$x = -\tfrac{1}{2}\,\frac{D + mE}{A + mB + m^2C},$$

but $m = \frac{y}{x}$, because the point is on the line $y = mx$. Substituting, etc., we have

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* The equation of this locus will take a simpler form if we introduce other constants in place of a, b and p. Thus, let c denote the side PA, d the side PB and a the included angle at P. Then $p^2 + a^2 = c^2$, $p^2 + b^2 = d^2$, $p\,(a + b) = cd \sin a$ (twice the area of the triangle), and $p^2 - ab = \frac{1}{2}\,[c^2 + d^2 - (a + b)^2] = cd \cos a$ (since by Trigonometry $(a + b)^2 = c^2 + d^2 - 2cd \cos a$), hence the equation becomes

$$c^2x^2 + d^2y^2 - 2cd \sin a\,.\,xy = c^2d^2 \cos^2 a.$$

The reader may discuss the cases $a = 0°$, $a = 180°$ and $a = 90°$.

$$Ax^2 + Bxy + Cy^2 + \tfrac{1}{2}Dx + \tfrac{1}{2}Ey = 0.$$

The required locus is therefore a similar conic passing through the fixed point, and by Art. 287, its centre is midway between that point and the centre of the given conic.

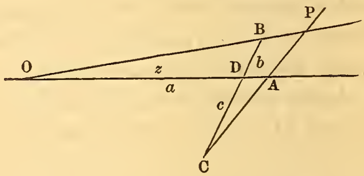
Intersection of Variable Lines.

337. The variable which we have to eliminate may be, as in some of the examples already given, one of the quantities which determine the position of a line whose equation we employ. Thus, in Art. 335, φ appears in the equation of CP. We have hitherto called such quantities *constants*, in distinction from x and y, because their values are determined by the position of a *fixed* line. When the line moves according to a certain law, these determining quantities become variable, as when we discuss the equation of a line, considering one of its constants arbitrary.

If now the locus required is described by the intersection of two moving or variable lines, it is necessary to express the equation of each of these lines in terms of constants and a single auxiliary variable. Then we shall have two equations between x, y and the auxiliary: the latter may therefore be eliminated. Thus:

A line of given length moves with one of its points in a fixed line with which it makes a constant angle; its extremities are joined by straight lines to two fixed points of the fixed line; find the locus of the intersection of the joining lines.

Take the fixed line as axis of X, one of the fixed points as origin, and the axis of Y parallel to the line of given length. Let b and c be the constant parts of this line above and below the axis, and a the distance OA between the fixed points. Let z represent the variable abscissa of D. Then the equation of OB passing through the origin and B (z, b) is $yz = bx$, and that of CA passing through



$(a, 0)$ and $(z, -c)$ is $y = \dfrac{-c}{z-a}(x-a)$, or $yz - ay = -cx + ac$.

Since P is on each of these lines it satisfies both their equations, therefore we have two relations between the co-ordinates of P and the auxiliary variable z. Eliminating the latter by subtraction,

$$ay = (b + c)x - ac.$$

The locus is therefore a straight line, passing through the points $(0, -c)$ and (a, b).

338. Instead of a single auxiliary, in expressing the equations of the variable lines, it is sometimes convenient to use two, between which the conditions of the problem give a relation. Thus, if we require the locus of the intersection of lines passing through the points $(a, 0)$ and $(-a, 0)$, the equations are $y = m(x - a)$ and $y = m'(x + a)$. Then, if the conditions give an equation between m and m', we may substitute the values $m = \frac{y}{x - a}$, $m' = \frac{y}{x + a}$, and the result will be the required locus. Supposing the axes rectangular, if the lines are to be perpendicular, the equation connecting m and m' is $mm' = -1$, and the locus will be found to be a circle. In general, if mm' is constant, the locus is a central conic of which $2a$ is an axis. If $m + m'$ is constant, the locus is always an hyperbola, and if $m - m$ is constant, it is a parabola.*

Examples.—Find the locus of the intersection of tangents to an ellipse at the extremities of conjugate diameters. (Since by Art. 195, the eccentric angles of the vertices differ by 90°, use the equations of the tangents in terms of the eccentric angles φ and $\varphi + 90°$, Art 192. Eliminate φ as in Art. 336.)

Find the locus of the intersections of perpendicular tangents of an ellipse. (We may use the form of Art. 190, or that of Art. 193; the locus will be found to be a circle as proved in Art. 194.)

Show that the locus of the intersection of a tangent to a conic, with a perpendicular from a focus, is a circle; but that of the intersection with a perpendicular from the centre is of the fourth degree, except when the conic is a circle.

A line parallel to the side BC of the triangle ABC, cuts AB in D, and AC in E, find the locus of the intersection of BE and CD.

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* Since m and m' are the tangents of the inclinations of the lines (axes being rectangular), we may find the loci when the sum or difference of the inclinations is constant, or when one is double the other, by the angular analysis.—See *Chauvenet's Plane Trig.*, Eqs. 123, 124, 137.

Point Connected with a Variable Line.

339. The equation of a variable line either contains one arbitrary constant, or constants connected together by such relations that all the rest may be expressed in terms of any one of them. For instance, the position of the straight line $y = mx + b$ is determined by two quantities m and b; when they are both regarded as arbitrary, the line is *wholly indeterminate*, but when they are connected by an equation, such as $b^2 = A^2m^2 + B^2$ (which is the condition of tangency to the ellipse, Art. 190), the line is *conditioned*, but not *determined*.* Other lines require more than two determining quantities, and are therefore *indeterminate* whenever these quantities are connected by a number of equations *less* than their own number; but the variable lines here considered are those in which the number of equations or conditions is *one less* than the number of quantities, so that one of them may be regarded as a variable of which all the rest are functions.

340. Now suppose that there is a point connected with a variable line; then, as the line varies, the point will describe a locus. Thus, the centre of a circle passing through two fixed points describes a locus while the circle varies. When the co-ordinates of the point enter as constants in the equation of the line, the conditions of the problem will give a relation between them which will be the equation required. For example:

To find the locus of the vertex of the parabola passing through P_1, *having a given parameter, and its axis parallel to the axis of* X.

The equation of the parabola is

$$(y - y')^2 = 2p\,(x - x'),$$

in which x' and y' are the co-ordinates of the vertex. By the conditions of the problem $2p$ *is constant*, and

$$(y_1 - y')^2 = 2p\,(x_1 - x').$$

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* Just as the position of a point, which is determined by two quantities x and y, is wholly indeterminate when they are both arbitrary; but when they are connected by a single equation the point is conditioned or restricted, though not determined.

This is a relation between x', y' and the constants $2p$, x_1 and y_1; hence if we regard x' and y' as variable co-ordinates, it is the equation of the locus of P'. Writing x and y in place of x' and y', we may put the equation in the form

$$(y - y_1)^2 = -2p(x - x_1).$$

Hence the locus is an equal parabola, extending in the opposite direction, and having P_1 for its principal vertex.

The number of conditions of the problem is one less than that which would determine the parabola. If there were another condition the vertex would be fixed; if there were one less, it would be free to take any position.

341. If instead of a constant value of $2p$, we have another condition, it will be necessary to determine $2p$ in terms of x', y' and constants, and substitute its value before changing x' and y' to x and y. For instance, *if the value of the parameter corresponding to the diameter passing through* P_1 *is to be constant*, instead of that of the principal parameter, we have by Arts. 152 and 147,

$$2p' = 2p(1 + n^2) = 2p + 4X_1 = 2p + 4(x_1 - x').$$

Hence $2p = 2p' - 4(x_1 - x')$, in which $2p'$ is constant; substituting this value in the equation of condition, we have

$$(y_1 - y')^2 = 2p'(x_1 - x') - 4(x_1 - x')^2.$$

Replacing x' and y' by x and y, the locus of the vertex is

$$(y - y_1)^2 + 4(x - x_1)^2 + 2p'(x - x_1) = 0,$$

an ellipse passing through P_1. Its equation when referred to P_1 as origin is $4x^2 + y^2 + 2p'x = 0$, from which it may be shown that the centre is the point $(-\frac{1}{4}p', 0)$ and the semi-axes are $A = \frac{1}{4}p'$, $B = \frac{1}{2}p'$, the major axis being perpendicular to the axis of X.

342. When the conditions of the problem give a relation between constants which appear in another form of the equation of the variable line, the equation must be reduced to that form, and the values of the constants substituted in the equation of condition. Thus:

Find the locus of the point whose polar relatively to the ellipse $A^2y^2 + B^2x = A^2B^2$ *is tangent to the ellipse* $A'^2y + B'^2x^2 = A'^2B'^2$.

The equation of the polar of P_1 is $A^2yy_1 + B^2xx_1 = A^2B^2$, and the condition of tangency for a line in the form $y = mx + b$, is (Art. 190) $b^2 = A'^2m^2 + B'^2$. Reducing the equation of the polar to the form $y = mx + b$, we find values of m and b, which we substitute in this equation of condition. The result is

$$\frac{B^4}{y_1^2} = A'^2 \frac{B^4x_1^2}{A^4y_1^2} + B'^2,$$

a relation between the co-ordinates of P_1, expressing the condition that its polar shall touch the ellipse $A'^2y^2 + B'^2x^2 = A'^2B'^2$. Putting x and y in place of these co-ordinates, and clearing of fractions,

$$A^4B^4 = A'^2B^4x^2 + B'^2A^4y^2.$$

The locus is therefore an ellipse whose semi-axes are $\frac{A^2}{A'}$ and $\frac{B^2}{B'}$; that is, third proportionals to the corresponding semi-axes of the given ellipse, and the ellipse of reference.

Examples.—Find the locus of the point whose polar passes through a given point.

Show that, when the polar of P_1 with reference to any conic touches an ellipse, the locus of P_1 is a conic. (Assume the conic of reference in the general form.)

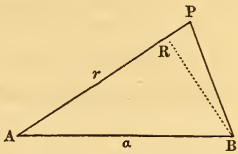
Show that the locus of the centre of the conic passing through the intersections of two conics is generally of the second degree; but that it is of the first degree when the conics are similar.

Use of Polar Co-ordinates.

343. Polar co-ordinates may be used with advantage when the point which describes the locus is connected with a fixed point by a line of variable length. For, if we make the fixed point the pole, and express the length of the connecting line (which will be the radius vector) in terms of its inclination, the result will be the polar equation. Thus:

A triangle has a fixed base; find the locus of its vertex, when the value of the vertical angle is given.

Let AB, the fixed base, be the initial line, and A the pole, then the side AP is the variable radius vector, and the angle at A is θ. Denote the constant value of the angle at P by α, and the distance AB by a. In order to express the value of r in terms of θ and the constants, drop a perpendicular from B on AP. It divides r into two parts, AR and PR, of which $AR = a \cos \theta$ and $PR = RB \cot \alpha = a \sin \theta \cot \alpha$. Therefore



$$r = a \cos \theta + a \cot \alpha \sin \theta.$$

Comparing this with the polar equation of Art. 112, we see that it represents a circle passing through the pole A (because $f = 0$), and having its centre at the point whose rectangular co-ordinates are $(\frac{1}{2}a, \frac{1}{2}a \cot \alpha)$.

344. We noticed in Art. 69, that the constants which enter into an equation of the form $x \cos \alpha + y \sin \alpha = p$ are the polar co-ordinates of the foot of the perpendicular from the origin. When the equation of a variable straight line is put in this form, we have, in place of p, its value in terms of α. Hence, if we write r in place of p and θ in place of α, we shall have the radius vector of the foot of the perpendicular expressed in terms of its angular co-ordinate θ; that is, the polar equation of the locus of this point.* An instance of this method of finding a locus is given in Art. 172, in which the locus of the foot of the perpendicular from the focus upon a tangent to the parabola is found. Applying the same method to the equation of Art. 171, we find for *the locus of the foot of a perpendicular from the vertex*

$$r \cos \theta = - \tfrac{1}{2}p \sin^2 \theta.$$

To ascertain the degree of this locus, it is necessary to transform it to rectangular co-ordinates by the formulæ of Arts. 86 and 97. Thus, multiplying both members by r^2 and transforming,

$$(x^2 + y^2)\, x + \tfrac{1}{2}\, py^2 = 0,$$

an equation of the third degree, and the same that would have been found, by using the equation of Art. 147 and that of a per-

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* In general, when p is a function of α, let $p = F.\alpha$, then the equation of the variable line is $x \cos \alpha + y \sin \alpha = F.\alpha$, and the locus of the foot of the perpendicular is $r = F.\theta$.

pendicular through the origin, and finding the locus of their intersection by the method of Art. 337.

345. If we put r equal to the perpendicular from any other point instead of the origin, the result will be the locus of the foot of the perpendicular from that point. Thus:

Find the locus of the foot of a perpendicular from (a, b) *on a tangent to an ellipse.*

The equation of the tangent, Art. 193, is

$$x \cos \alpha + y \sin \alpha = \sqrt{A^2 \cos^2 \alpha + B^2 \sin^2 \alpha}.$$

The perpendicular from (a, b) is

$$\sqrt{A^2 \cos^2 \alpha + B^2 \sin^2 \alpha} - a \cos \alpha - b \sin \alpha.$$

Putting r equal to this quantity, and writing θ in place of α,

$$r = \sqrt{A^2 \cos^2 \theta + B^2 \sin^2 \theta} - a \cos \theta - b \sin \theta,$$

or $$\sqrt{A^2 \cos^2 \theta + B^2 \sin^2 \theta} = r + a \cos \theta + b \sin \theta.$$

Multiplying through by r and transforming,

$$\sqrt{A^2x^2 + B^2y^2} = x^2 + y^2 + ax + by.*$$

Squaring this equation, we shall find it to be, in general, of the fourth degree: but it will be divisible by $x^2 + y^2$, when $a^2 = c^2$ and $b = 0$; that is, when the point is one of the foci. In this case, but in no other, will it reduce to the second degree.

EXAMPLES.—A straight line passing through a fixed point A on the circumference of a circle meets the circle again in B, find the locus of a point P moving in the line, so that $AB \times AP = c^2$.

Find the locus when A is not on the circumference. (Take the diameter through A as initial line, then the value of AB is one of the values of r in Art. 114.)

Find the locus of the foot of a perpendicular from the origin upon the line passing through P′.

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* This is the general equation of the curve, referred to the point (a, b) itself as origin. Since it is satisfied by $x = 0$, $y = 0$, it would seem that the curve always passed through the point, which is impossible when the point is within the ellipse. The reason is that we introduced the root $r = 0$ by multiplying through by r.

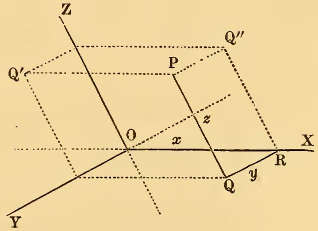
CHAPTER X.

APPLICATION OF ANALYSIS TO SOLID GEOMETRY.

346. We observed in Art. 6, that to determine the position of a point on a given surface two co-ordinates or determining quantities are necessary and sufficient; but that to determine position generally *three* are necessary on account of the three dimensions of space. The investigation of questions involving the position of points not all in the same plane is therefore called Geometry of Three Dimensions, or *Solid Geometry*, because the discussion of solids requires the consideration of the three dimensions, while that of plane figures involves only two.

It is the object of the present Chapter to explain a method by which the analytical treatment is adapted to Solid Geometry. The systems of co-ordinates used are extensions of those we have already applied to Plane Geometry.

347. Let OX and OY be two co-ordinate axes taken in a fixed plane of reference; the position of any point in this plane is then determined by the co-ordinates x and y. Through the origin let a line OZ be drawn, not in the plane YOX. If now from any point P a line be drawn parallel to OZ and piercing the plane YOX in Q, the length of PQ, together with the co-ordinates of Q as referred to the axes OX and OY, determines the position of P. Therefore, denoting PQ by z, QR by y, and OR by x, we may regard P as determined by the values of three co-ordinates, x, y and z, which are distances measured in the directions of three fixed lines or axes, OX, OY and OZ, called respectively, *the axis of* X, *the axis of* Y *and the axis of* Z.



If from P a line be drawn parallel to the axis of X, meeting the plane YOZ in Q′, PQ′ and OR will be equal, because they are parts of parallel lines intercepted between the parallel planes PQR and YOZ. Therefore x represents PQ′, the distance of P from the plane YOZ, measured in a direction parallel to the axis of X. If a line be drawn parallel to the axis of Y to meet the plane ZOX in Q″, PQ″ and QR will be equal, and each may be represented by y; therefore the co-ordinates x, y and z may be considered to denote the distances of P from three *co-ordinates planes*, measured from each in a direction parallel to the intersection of the other two. These planes are called respectively *the plane of* XY, *the plane of* YZ and *the plane of* XZ. The first contains the axis of X and that of Y, the second that of Y and that of Z, the third that of X and that of Z.

348. The plane of the lines PQ′ and PQ″ is parallel to the plane of XY, that of PQ and PQ″ to the plane of YZ, and that of PQ and PQ′ to the plane of XZ. The intersections of these planes and the co-ordinates planes, therefore, form the edges of a parallelopipedon, or solid whose faces are parallelograms. For every point in the first of these planes, the value of z is the same; if this constant value be denoted by c, then the equation $z = c$ indicates that the point P is situated in this plane. It is, therefore, said to be the *equation* of the plane. In like manner, the other planes are represented by equations of the form $x = a$ and $y = b$. The point whose co-ordinates are a, b and c, or as we may express it the point (a, b, c), is therefore the intersection of the three planes $x = a$, $y = b$, $z = c$, each of which is parallel to one of the co-ordinate planes.

In constructing a point with given co-ordinates, it is evident that we may lay off the values a, b and c in their proper directions, and in any order we please. Thus, we may lay off the value c on the axis of Z, and so determine the plane $z = c$; then in this plane we may construct the point (a, b) using as axes its intersections with the planes of XZ and of YZ; that is, the lines OQ″ and OQ′, which are parallel to the original axes of X and Y.

It is of course necessary to assume a positive direction on the axis of Z, as well as on the axis of X and on that of Y. In representing position in space it is usual to draw only the positive directions of the axes, and to indicate the position of a point, by drawing the co-ordinates PQ and QR parallel to the axes of Z and Y.

Co-ordinates of Direction.

349. When the position of a point in a given plane is determined by its distance and direction from a fixed point, only one angular co-ordinate is necessary, because we have only to compare the directions of lines in a single plane. But for points in space *two* co-ordinates of direction are necessary; one to determine a plane containing the line whose direction is to be expressed, and the other to fix the direction of the line in that plane.

Let O be the fixed point of reference, and OZ a fixed straight line passing through it. It will be convenient to conceive of this line as directed upward from O, or toward the zenith. A plane passing through OZ will then be a *vertical* plane. Suppose such a plane to rotate about OZ as an axis, and let P be the point whose position is to be determined. It is evident that the vertical plane may be made to pass through P, and therefore to contain the line OP.

Let OH be a perpendicular to OZ, drawn in the vertical plane; then in the rotation OH will describe a horizontal plane. Assume an initial line in the horizontal plane, then the direction of OH in this plane may be expressed, as in the system of polar co-ordinates, by θ, denoting its inclination to the initial line. The angle θ may therefore be used to determine the position of the vertical plane containing OP. Let φ denote the inclination of OP to OH measured in the vertical plane; then the angles θ and φ together determine the direction of OP.

350. The co-ordinates of direction, θ and φ, are usually called *spherical co-ordinates*, because it is found convenient, in the treatment of questions where distance is not considered, to refer *direction in space* to position upon the surface of a sphere. Thus, suppose the point O, of the last Article, to be the centre of a sphere of any radius we chose; the line OP, produced if necessary, will pierce the sphere in some point, and the point P is said to be *projected* on the surface of the sphere at that point. The position of this point is then used to express the direction of OP, or apparent position of P as seen from O. The horizontal plane described by OH will cut the sphere in a great circle, which is *the primary circle* of reference. The axis OZ pierces the surface of the sphere in

points which are called *the poles* of this circle. Now it is evident that θ is equivalent to an arc of the primary circle measured from the point in which the sphere is pierced by the initial line, which is *the origin* of the spherical co-ordinates. The vertical plane cuts the sphere in a great circle passing through the poles of the primary circle, and φ is measured by the arc of this circle included between the projection of P and the extremity of the arc θ.

351. Let the angle φ be measured upward. If the value of θ is fixed, $\varphi = 0$ corresponds to a point in the horizontal plane; if φ increase from zero, the line OP moves upward, its inclination to the horizontal plane increasing, until $\varphi = 90°$, when it becomes perpendicular to that plane and coincides with the vertical line OZ. If φ increase beyond $90°$, the inclination of OP to the horizontal plane is diminished, being measured by $180° - \varphi$; but if we add $180°$ to the value of θ the direction of OP will be expressed by a value of φ less than $90°$. In other words, $180° + \theta$ and $180° - \varphi$ determine the same direction as θ and φ; hence if we allow θ all values between $0°$ and $360°$, φ may always be taken less than $90°$. For a point below the horizontal plane, φ is negative; and $\varphi = -90°$ makes OP coincide with OZ produced, whatever be the value of θ. The limiting values of φ are therefore $+ 90°$ and $- 90°$, and these extreme values of φ determine the direction of a line without the aid of the co-ordinate θ.

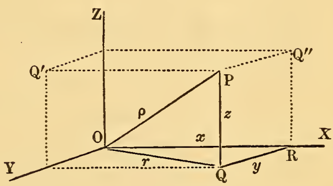
The length of OP, together with φ and θ which determine its direction, constitutes a system of *polar co-ordinates* for space. The plane in which θ is measured is called *the primitive plane.* The line OZ perpendicular to the primitive plane is called its *axis.* The initial line may be taken anywhere in the primitive plane; its co-ordinates of direction are $\theta = 0°$ and $\varphi = 0°$. Any two straight lines which meet at right angles may be made the axis and initial line of a system of polar co-ordinates.

Polar and Rectangular Co-ordinates.

352. If the primitive plane of polar co-ordinates be taken as the plane of XY, and the axis or perpendicular OZ, as the axis of Z, then the co-ordinate z will be a perpendicular from P to this plane. Let Q be the foot of the perpendicular. This point is called the *projection* of P on the plane, and the straight line OQ, joining Q

with the origin, is called the projection of OP. We may then say that φ is the angle between OP and its projection on the primitive plane, and that θ is the angle between this projection and the initial line. Denoting the length OQ by r, r and θ are the polar co-ordinates of Q in the primitive plane. If now rectangular axes be assumed in this plane, the axis of X coinciding with the initial line, it is evident that each of the axes is perpendicular to the plane of the other two. In this case, the axes of X, Y and Z are said to form a *rectangular system*. This system is universally used in the applications of Analysis to Mechanics and Astronomy, on account of its connection with the polar system.

353. In the figure, ROQ is the angle θ, and QOP is the angle φ. Denote the distance OP by ρ; we have now to find the relations between the polar co-ordinates ρ, φ and θ, and the rectangular co-ordinates x, y and z.



By the relations between the rectangular and polar co-ordinates of Q in the plane of XY, we have

$$x = r \cos \theta \qquad \text{and} \qquad y = r \sin \theta.$$

In the plane ZOPQ, ρ and φ may be considered as the polar co-ordinates of P; while r and z are its rectangular co-ordinates, because PQO is a right angle. Therefore

$$r = \rho \cos \varphi \qquad \text{and} \qquad z = \rho \sin \varphi.$$

Hence, eliminating r from the values of x and y, we find

$$x = \rho \cos \varphi \cos \theta,$$
$$y = \rho \cos \varphi \sin \theta,$$
$$z = \rho \sin \varphi.$$

354. If through P planes be passed parallel to the three co-ordinate planes, the intersections will form the edges of a rectangular parallelopipedon as represented by the dotted lines in the figure. The edges PQ, PQ′ and PQ″ are the distances of P from the planes of XY, YZ and XZ respectively, because they are perpendiculars to these planes. Therefore the co-ordinates x, y, z, in the rectangular

system, are the distances of P from the co-ordinate planes. The diagonal, OQ, which we have denoted by r is the distance of P from the axis of Z. By the right triangle QOR, we have $r^2 = x^2 + y^2$. In like manner, if we denote OQ′ and OQ″ by r' and r'', $r'^2 = y^2 + z^2$ and $r''^2 = z^2 + x^2$. Therefore the distances of P from the axes of X, Y and Z respectively are

$$\sqrt{y^2 + z^2}, \qquad \sqrt{z^2 + x^2}, \qquad \sqrt{x^2 + y^2}.$$

The right triangle POQ gives $\rho^2 = r^2 + z^2$, therefore for the distance of P from the origin, we have

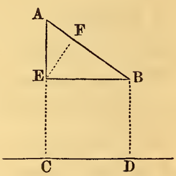
$$\rho = \sqrt{x^2 + y^2 + z^2}.$$

Example.—Find the distance of the point (3, 4, 12) from each axis and from the origin.

Method of Projections.

355. The relations between points and lines in space are most conveniently established by the aid of the method of projections, which is now to be explained.

A point is said to be *projected* on a straight line at the foot of a perpendicular from the point to the straight line. The distance between the points in which the extremities of a given line are projected on a fixed line of indefinite length, is called the *projection* of the given line upon the indefinite line. Thus, CD is the projection of AB upon the horizontal line in the figure. If AB and the line on which it is projected are in the same plane, then the projecting perpendiculars, AC and BD, are also in this plane. Draw BE parallel to CD, then ABE is the inclination of AB to the indefinite line, and EB is equal to its projection. Now in the right triangle ABE,



$$EB = AB \cos ABE;$$

that is, *the projection of a line is equal to its length multiplied by the cosine of its inclination to the line on which the projection is made.*

The projection of a line cannot be greater than the line itself, because the cosine of an angle cannot exceed unity. The projection upon a parallel line is equal to the line itself, and the projection upon a perpendicular line is zero.

356. If EF be drawn perpendicular to AB, BF is the projection of EB on AB; therefore $BF = EB \cos ABE = AB \cos^2 ABE$. Now AE is the projection of AB upon AC, and AF is the projection of AE upon AB; hence $AE = AB \cos BAE$, and $AF = AB \cos^2 BAE$. Adding the values of BF and AF, we have $AB = BF + AF = AB (\cos^2 ABE + \cos^2 BAE)$; therefore $\cos^2 ABE + \cos^2 BAE = 1$. Since the angles ABE and BAE are complements, the above is a method of proving the fundamental equation of trigonometry, $\sin^2 + \cos^2 = 1$. If we multiply both members by AB^2, we have (by substituting EB for AB cos ABE, and AE for AB cos BAE) $EB^2 + AE^2 = AB^2$, the fundamental relation between the sides of a right triangle. This relation may be expressed thus: If a line in a given plane be projected upon two perpendicular lines of the plane, *the sum of the squares of the projections will equal the square of the line.*

357. The projection of AB upon any line, whether in the same plane with it or not, may be made by passing planes perpendicular to this line through A and B. The part of the line intercepted between the planes will be the projection. The projections of a line upon two parallel lines are equal, each being the distance between the same two parallel planes. Since all parallel lines have the same direction, a line is considered as having the same inclination to a line *which does not intersect it* as to a parallel line which does intersect it. Hence the projection of a line is in all cases equal to its length multiplied by the cosine of its inclination.

The projections of equal and parallel lines upon the same line are evidently equal.

358. Suppose now the projections to be made upon a fixed line, and let one direction, measured upon the fixed line, be regarded as positive. Let any two points, A and B, be joined by a straight line, and also by a broken line ACDB, the intermediate points C and D having any positions whatever. Now, if by the projection of AB we mean the distance *from* the projection of A *to* that of B, considering both the length and direction of this distance, then the projection of AB may be positive, or it may be negative. It will

be negative, when the direction from A to B makes an obtuse angle with the positive direction assumed on the fixed line. The ratio of a line to its projection will still be that of unity to the cosine of this angle, because the cosine of an obtuse angle is negative.

If now we consider the signs of the projections, it is easy to see that *the projection of the straight line* AB *is equal to the sum of the projections of the parts of the broken line* ACDB; that is, of AC, CD and DB.

359. A point is said to be projected on a plane at the foot of a perpendicular from the point to the plane. If all the points of a given straight line be projected upon a plane, the projecting perpendiculars will lie in a single plane. This plane may be called *the projecting plane* of the given line; it is perpendicular to the plane on which the projection is made, and cuts it in a line which is called *the projection* of the given line. Therefore the acute angle between a line and its projection is the inclination of the line to the plane of projection. The projection of a line of definite length upon a plane is therefore equal to its length multiplied by the cosine of its inclination.

The projections of a given line upon a perpendicular line and plane are in reality projections upon two perpendicular lines, because a line perpendicular to a plane is perpendicular to every line in the plane. Therefore the sum of the squares of these projections is equal to the square of the line, by Art. 356.

Direction Angles.

360. According to the preceding definitions, the co-ordinates x, y and z in the rectangular system are the projections of OP or ρ, the distance of P from the origin, upon the axes of X, Y and Z respectively. For the plane PQR, in the figure, is perpendicular to the axis of X, therefore the distance OR cut off on the axis is the projection of OP; and in like manner the other co-ordinates are equal to distances cut off on their respective axes by perpendicular planes passing through P.

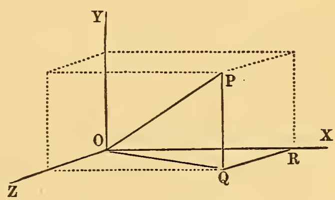
Let α denote the angle between OP and the positive direction of the axis of X, let β denote that between OP and OY, and γ that between OP and OZ. Then, regarding ρ as positive, we shall have

$$x = \rho \cos \alpha, \qquad y = \rho \cos \beta, \qquad z = \rho \cos \gamma.$$

The direction of the line OP passing through the origin is determined by the values of these angles; because, if they are known, we may assume a value for ρ and then determine a point on the line, which with the origin fixes the line's direction.

We saw in Art. 349, that two angles are sufficient to determine the direction of a line in space. The direction of a given line might be defined by means of two angles—namely, its inclination to its projection in the plane of XY and the inclination of that projection to the axis of X. For a line passing through the origin, as OP, these angles would correspond to the polar co-ordinates, φ and θ, of a point of the line. But more symmetrical results are obtained by introducing into our equations α, β and γ, as *the direction angles* of a given line, although they are not three independent quantities, but are connected by a relation which we are now to find.

361. The co-ordinates of the point P, in the figure, OR, RQ and QP, form a broken line joining the points O and P; therefore, as explained in Art. 358, the sum of their projections upon any line is equal to the projection of OP on the same line. The sum of the projections of the co-ordinates upon the line OP is therefore equal to OP or ρ. The projection of x upon OP is $x \cos \alpha$, because α is its inclination to OP. Hence, the value of this projection is $\rho \cos^2 \alpha$. In like manner, the projection of y is $\rho \cos^2 \beta$, and that of z is $\rho \cos^2 \gamma$. Hence $\rho \cos^2 \alpha + \rho \cos^2 \beta + \rho \cos^2 \gamma = \rho$, or



$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

This is the fundamental relation between the three direction angles α, β and γ.* If we multiply each member by ρ^2, and put

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* These direction angles may be represented by arcs of great circles upon the surface of a sphere, in the following manner: The origin being the centre of the sphere, let X, Y and Z be the points in which the positive axes pierce the surface; then XYZ is a tri-rectangular and tri-quadrantal triangle; that is, its angles are all right angles, and its sides quadrants.

x, y and z in place of their values, we obtain the equation

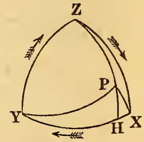
$$x^2 + y^2 + z^2 = \rho^2;$$

that is, *the sum of the squares of the projections of a line upon three perpendicular lines equals the square of the line.*

We have thus found by the method of projections the relation between the distance from the origin and the rectangular co-ordinates of a point, that was found in Art. 354.

362. The quantities $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called *direction cosines.* They are the factors by which we multiply the length of a line to produce its projections upon the axes, and they may have any values, of which the sum of the squares is unity. Thus, $\frac{2}{7}$, $\frac{3}{7}$ and $\frac{6}{7}$ are the direction cosines of a certain line. It is not necessary to attribute them to a line passing through the origin, if, as explained in Art. 357, parallel lines are considered as having the same inclinations to the axes. If a line makes an obtuse angle with one of the axes, the corresponding direction cosine is negative. It is evidently only necessary to consider values of the direction angles less than 180°.

The arcs PX, PY and PZ will then measure the direction angles, α, β and γ, of the radius OP.



Produce ZP to meet XY in H; then because Z is the pole of XY, ZP is the complement of PH, and $\cos^2 ZP + \cos^2 PH = 1$. For a similar reason, $\cos^2 XH + \cos^2 HY = 1$, therefore $\cos^2 ZP + \cos^2 PH\,(\cos^2 XH + \cos^2 HY) = 1$. But the arc PH is perpendicular to XY, hence, by a formula of spherical right triangles, $\cos PH \cos XH = \cos PX$ and $\cos PH \cos HY = \cos PY$. Therefore by substitution, $\cos^2 ZP + \cos^2 PX + \cos^2 PY = 1$, which is the above fundamental formula.

The arcs PH and XH are the spherical co-ordinates ϕ and θ, of the point P, or the angular co-ordinates of the direction of OP. (See Art. 350.) Comparing the values of x in Arts. 353 and 360, we have $\cos\phi \cos\theta = \cos\alpha$, which is a proof of the formula of spherical trigonometry above referred to. The axes in the figures are so taken, that increase in the angle θ (which is measured from the axis of X toward that of Y) corresponds to *positive rotation* about the axis of Z. If the axis of Z is directed *upward*, this rotation takes place in a horizontal plane, and by common consent it is assumed to be positive, when in the order of the cardinal points of the compass N.E.S.W., or in the direction of the hands of a watch. The arrows indicate the direction of positive rotation about each of the axes.

If one of the direction angles is $0°$, the line is parallel to the corresponding axis, and each of the other direction angles is $90°$. Thus, if $\alpha = 0°$, it is parallel to the axis of X, therefore $\beta = 90°$ and $\gamma = 90°$; the direction cosines in that case are $\cos\alpha = 1$, $\cos\beta = 0$, $\cos\gamma = 0$. The values $\alpha = 180°$, $\beta = 90°$, $\gamma = 90°$, or $\cos\alpha = -1$, $\cos\beta = 0$, $\cos\gamma = 0$, correspond to the negative direction of the axis of X. If one of the direction angles is $90°$, the line is in a plane perpendicular to the corresponding axis. Thus, if $\gamma = 90°$, it is in a plane parallel to that of XY.

If two of the direction cosines of a line and the sign of the third are known, the direction is determined. Thus, given $\cos\alpha = \frac{2}{3}$ and $\cos\beta = \frac{1}{3}$; by the fundamental formula we obtain $\cos^2\gamma = \frac{4}{9}$, hence $\cos\gamma = \pm\frac{2}{3}$. Therefore the direction is not determined unless we know the sign of $\cos\gamma$. If we change the signs of all three of the direction cosines at once, we obtain those which belong to the opposite direction. Thus, $\frac{2}{7}$, $-\frac{3}{7}$ and $\frac{6}{7}$ indicate one direction in a certain line, and $-\frac{2}{7}$, $\frac{3}{7}$, $-\frac{6}{7}$ indicate the opposite direction in the same, or in a parallel line.

363. *To find the angle between two lines of which the direction cosines are given.*

Let α, β and γ denote the direction angles of one of the lines, and α', β' and γ', those of the other. Let δ denote the mutual inclination of the lines, which we will at first suppose to pass through the origin. Let ρ denote any length OP measured off from the origin on the first line, then the co-ordinates of P are by Art. 360, $x = \rho\cos\alpha, y = \rho\cos\beta$, $z = \rho\cos\gamma$. The projection of ρ upon the second line is the sum of the projections of x, y and z (Art. 358). Since δ is the angle between the lines, the projection of ρ is $\rho\cos\delta$. Since α' is the inclination of the axis of X to the second line, the projection of x is $x\cos\alpha'$. In like manner, the projection of y is $y\cos\beta'$, and that of z is $z\cos\gamma'$. Therefore

$$\rho\cos\delta = x\cos\alpha' + y\cos\beta' + z\cos\gamma'.$$

Substituting the values of x, y and z, and dividing by ρ, we obtain

$$\cos\delta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

Since parallel lines have the same direction angles, this result applies to lines which do not pass through the origin, as well as to those which do. The value of δ may be found from its cosine by means of the trigonometric tables.

Examples.—Find the cosine of the angle between the lines whose direction cosines are respectively $\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$, and $\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$. (The direction cosines are always given in the order $\alpha\beta\gamma$.)

Find two values of the third direction cosine, the values of the first two being $\frac{7}{9}$ and $\frac{4}{9}$; and find the cosine of the angle between the resulting directions.

Show that the formula gives the supplemental angle when one of the directions is reversed; also that, if the directions are the same, the result is $\delta = 0°$.

What is the value of $\cos\delta$, when both the given lines are in the plane of XY? (For a line in this plane $\cos\gamma = 0$, therefore by the fundamental formula $\cos^2\alpha + \cos^2\beta = 1$ or $\cos\beta = \pm\sin\alpha$.*)

What is the value of $\cos\delta$, when one of the lines is in the plane of XY and the other in that of XZ?†

364. If $\delta = 90°$, $\cos\delta = 0$; therefore *the condition that two lines shall be perpendicular is*

$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = 0.$$

Let a, b and c represent the direction cosines of a known line, and q, r, s, those of an unknown line; then, by the fundamental formula of Art. 361, we have the equation $q^2 + r^2 + s^2 = 1$, and if the line is to be perpendicular to the given line, $aq + br + cs = 0$. We have, therefore, but two equations between three unknown quantities. Since three equations will determine the values of q, r and s, a line may be found perpendicular to two given lines. For example, let the direction cosines of one of the given lines be $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$, and those of the other $\frac{14}{15}, -\frac{1}{3}, \frac{2}{15}$. The three equations to be solved are

$$q^2 + r^2 + s^2 = 1,$$
$$2q + r + 2s = 0,$$

and

$$14q - 5r + 2s = 0.$$

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* The result is the formula of plane trigonometry for the difference or for the sum of two angles, according as α and α' are measured in the same or in opposite directions.

† The result is a formula of spherical right triangles, since the planes are perpendicular.

From the two equations of the first degree, we obtain by elimination, $r = 2q$ and $s = -2q$. Substituting these values in the quadratic equation, we find $q = \pm \frac{1}{3}$; therefore $r = \pm \frac{2}{3}$ and $s = \mp \frac{2}{3}$. The problem has two solutions, which, however, only express the two opposite directions in the same straight line.

EXAMPLES.—Show that $\frac{9}{11}$, $-\frac{6}{11}$, $\frac{2}{11}$ and $\frac{6}{11}$, $\frac{7}{11}$, $-\frac{6}{11}$ are the direction cosines of two perpendicular lines; and find those of a line perpendicular to both.

Given $-\frac{6}{7}$, $\frac{3}{7}$, $\frac{2}{7}$ and $\frac{2}{7}$, $\frac{6}{7}$, $-\frac{3}{7}$, show that the lines are perpendicular, and find a line perpendicular to both.

TRANSFORMATION OF CO-ORDINATES.

365. In passing from one system of co-ordinate axes to another, we may consider separately the case in which the origin is changed and that in which the directions of the axes are changed.

Let P′, the point whose co-ordinates are x', y', z', be the origin of a new system of co-ordinates, in which the directions of the axes are unchanged. The new plane of XY is parallel to the old, and will divide the co-ordinate z of any point into two parts, one of which is z', and the other is the corresponding new co-ordinate of the point. Denoting this by Z we have $z = z' + \mathrm{Z}$. Treating the other co-ordinates in the same manner, we obtain, as extensions of the formulæ of Art. 85,

$$x = \mathrm{X} + x', \qquad y = \mathrm{Y} + y', \qquad z = \mathrm{Z} + z'.$$

We may also express the new co-ordinates in terms of the old, thus:

$$\mathrm{X} = x - x', \qquad \mathrm{Y} = y - y', \qquad \mathrm{Z} = z - z'.$$

Of course the values of the constants x', y' and z' may be negative, as well as those of the variables, these expressions denoting algebraic sums and differences.

EXAMPLES.—What does the equation $x^2 + y^2 + z^2 = 81$ become, when the origin is placed at the point $(4, 7, -4)$?

What are the new co-ordinates of $(1, -3, -2)$, when referred to the origin $(-2, 2, -1)$?

366. Now suppose the old axes to be rectangular and the origin to be unchanged. Let a, b and c be the direction cosines of the

new axis of X; a', b', c', those of the new axis of Y; and a'', b'', c'', those of the new axis of Z. Denote the new co-ordinates of the point P by X, Y and Z, and the old co-ordinates by x, y and z.

Now, whether the new axes be rectangular or oblique, the co-ordinates X, Y and Z form a broken line connecting P with the origin (see Fig. Art. 347). Therefore the projection of OP upon any line is equal to the sum of the projections of X, Y and Z. Since the old axes are rectangular, x is the projection of OP upon the old axis of X. Since a is the cosine of the inclination of the new axis of X to the old, the projection of the co-ordinate X is aX. In like manner, a' being the cosine of the inclination of the new axis of Y to the old axis of X, the projection of Y is a'Y. The projection of Z is, for the same reason, a''Z; hence $x = a\mathrm{X} + a'\mathrm{Y} + a''\mathrm{Z}$. Proceeding in the same manner, we obtain expressions for y and z, the projections of OP upon the old axes of Y and Z respectively; the projecting ratios being in the first case b, b' and b'', in the second, c, c' and c''. Hence the formulæ,

$$x = a\mathrm{X} + a'\mathrm{Y} + a''\mathrm{Z},$$
$$y = b\mathrm{X} + b'\mathrm{Y} + b''\mathrm{Z},$$
$$z = c\mathrm{X} + c'\mathrm{Y} + c''\mathrm{Z}.$$

The formulæ contain nine constants, but they are not all independent, for, since a, b and c are direction cosines, $a^2 + b^2 + c^2 = 1$, and, for the same reason, $a'^2 + b'^2 + c'^2 = 1$ and $a''^2 + b''^2 + c''^2 = 1$. Hence the nine constants are connected by three equations, and only six of them can be regarded as independent.

367. If the new axes of X and Y are perpendicular, we shall have, by Art. 364, the equation $aa' + bb' + cc' = 0$ between their direction cosines. Similar equations will express that the new axis of Z is perpendicular to that of X, and to that of Y. Therefore, *if the new axes, as well as the old, are rectangular*, the nine constants will be connected by the following six relations:

$$\begin{aligned} a^2 + b^2 + c^2 &= 1, & a a' + b b' + c c' &= 0, \\ a'^2 + b'^2 + c'^2 &= 1, & a a'' + b b'' + c c'' &= 0, \\ a''^2 + b''^2 + c''^2 &= 1, & a' a'' + b' b'' + c' c'' &= 0. \end{aligned}$$

Only three of the nine constants can in this case be regarded as independent.* If we should assume values for a, b and a', for instance, the other six might be determined; but there would be several solutions, because three of the equations are of the second degree.

368. Since a, a' and a'' are the cosines of the inclinations of the three new axes to the old axis of X, they are its direction cosines *as referred to the new axes.* So also b, b' and b'' are the direction cosines of the old axis of Y, and c, c', c'' those of the old axis of Z. Hence, the new rectangular co ordinates expressed in terms of the old, are

$$X = a\,x + b\,y + c\,z.$$
$$Y = a'\,x + b'\,y + c'\,z,$$
$$Z = a''x + b''y + c''z.$$

We may also prove, in the same way as in the last Article, the six relations,

$$a^2 + a'^2 + a''^2 = 1, \qquad ab + a'b' + a''b'' = 0,$$
$$b^2 + b'^2 + b''^2 = 1, \qquad ac + a'c' + a''c'' = 0,$$
$$c^2 + c'^2 + c''^2 = 1, \qquad bc + b'c' + b''c'' = 0.$$

These new relations between the nine constants are of course consequences of the six relations first found.

Examples.—Show that, if the direction cosines of the new axis of X are $\frac{2}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, of Y, $-\frac{1}{3}$, $\frac{14}{15}$, $-\frac{2}{15}$, and of Z, $-\frac{2}{3}$, $-\frac{2}{15}$, $\frac{11}{15}$, the new axes will be rectangular. Find in this system the new co-ordinates of the point $(1, 2, -1)$.

What does the equation $x^2 + y^2 + z^2 = R^2$ become, when transformed to a new system of rectangular co-ordinates?

Verify the value of X in Art. 368, by means of the formulæ of Art. 366 and the equations of Art. 367.

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* Formulæ for transformation from one system of rectangular co-ordinates to another are frequently given, in which the constants employed are the functions of three independent angles determining the position of the new planes and axes. But it is impossible so to select these angles as to avoid complicated and unsymmetrical results. In a practical case the several direction cosines are readily determined, and in the general discussion of equations the greatest advantage is gained by the use of the formulæ of Art. 366, and the twelve symmetrical relations between the constants in Arts. 367 and 368.

369. If two systems of oblique axes have the same origin, the rectangular co-ordinates of a point may be expressed in terms of its co-ordinates in each of the oblique systems, by means of the formulæ of Art. 366. The equality of the values of each of the rectangular co-ordinates gives an equation of the first degree between the co-ordinates in the two oblique systems. Solving the three equations thus found, we might express the co-ordinates of one system in terms of those of the other system, and the expressions found would be of the first degree. Therefore the general formulæ of transformation for change in the direction of the axes would be of the same form as the equations of Art. 366, but the coefficients a, b, c, etc., would have different significations. Now these formulæ, and also those of Art. 365, being of the first degree, it is evident that transformation of co-ordinates cannot raise the degree of an equation between x, y and z. Neither can it lower the degree, because the reverse transformation, which must reproduce the original equation, cannot raise the degree.

The formulæ for transformation from rectangular to polar co-ordinates have been already found in Art 353.* To express the polar co-ordinates in terms of the rectangular, we have, from the values of x, y, z and r (since $r = \sqrt{x^2 + y^2}$),

$$\tan\theta = \frac{y}{x}, \qquad \tan\varphi = \frac{z}{\sqrt{x^2 + y^2}},$$

and

$$\rho = \sqrt{x^2 + y^2 + z^2}.$$

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* In passing from one system of polar co-ordinates to another, the polar values of the old and new co-ordinates may be substituted in the formulæ for change of origin, or in those for change of direction of the axes. In the latter case, the value of ρ is unchanged, and the resulting equations become (by dividing through by ρ) relations between two systems of spherical co-ordinates. If the primitive planes coincide, the value of ϕ is unchanged, and θ is measured from a new initial line, therefore the difference of the two values of θ (or their sum, if measured in opposite directions) will be known. If the planes of XZ coincide, the value of y is unchanged, and the relations between the old and new values of x and z may be found by the formulæ of Art. 88 for rectangular transformation in a plane. A similar method may be used when the planes of YZ coincide. These planes may always be made to coincide, by taking the intersection of the primitive planes as the initial line or axis of X.

Equations between Co-ordinates.

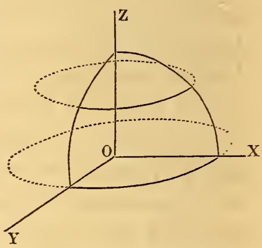
370. The position of a point in space is determined by the values of three co-ordinates; and since the values of three unknown quantities may be determined by three equations, the position of a point is in general determined by three equations between its co-ordinates x, y and z. Let us now consider the meaning of a single equation connecting these quantities.

It is evident that x, y and z, in a given equation, may take a variety of values; those of x and y, for instance, may be assigned at pleasure, and a corresponding value of z may be derived from the equation. Thus, given the equation $2x - 3y + z = 5$; if we assume $x = 2$ and $y = 1$, the equation gives $z = 4$; if we assume $x = 1$ and $y = 0$, it gives $z = 3$. We may therefore regard z as a *function of two independent variables*, x and y, because it depends upon them both for its value. When the value of the function is directly expressed in terms of the independent variables, it is called an explicit function. Thus, from the above equation we derive $z = 5 - 2x + 3y$, in which z is made an explicit function of x and y.

By substituting any assumed values for x and y, a corresponding value of z may be obtained, and thus we may determine and construct any number of points whose co-ordinates satisfy the given equation.

371. To obtain an idea of the situation of the various points which satisfy a given equation, we must consider first those which have a fixed value for one co-ordinate. We have seen in Art. 348 that the point (a, b, c) may be constructed by constructing the point (a, b) in the plane $z = c$, which is parallel to the plane of XY. Now if we give to z, in the equation, a certain value, the result is an equation between the co-ordinates x and y of all the points situated in a certain plane which satisfy the original equation. Thus, given the equation $x^2 + y^2 + z^2 = 25$; if we make $z = 3$, we have $x^2 + y^2 = 16$. Supposing the axes rectangular, this is the equation of a circle whose radius is 4. Therefore a circle having this radius, constructed in the plane $z = 3$, will contain all the points of that plane which satisfy the given equation.

In general, if $z = c$, we have the equation $x^2 + y^2 = 25 - c^2$, representing a circle in the plane $z=c$, whose centre is in the axis of Z. The radius of the circle is $\sqrt{25-c^2}$. If $c=0$, the plane coincides with the plane of XY, and the radius of the circle is 5; and if we suppose c to increase from this value, the plane moves upward, continuing parallel to the plane of XY. The radius of the circle decreases gradually as the plane moves, and its circumference will describe a surface. When $c=5$, the radius of the circle becomes zero; if c increase beyond that value, the circle becomes imaginary. If c be made negative, the radius of the circle will decrease in the same manner as we pass from $c=0$ to $c=-5$, and beyond this limit it is again imaginary. It is plain that the surface thus described will contain all the points which satisfy the equation $x^2+y^2+z^2=25$.



372. It may be shown in like manner of any other equation containing x, y and z, that all the points which satisfy it are situated upon a certain surface. The curves of which the equations are found by giving fixed values to z, are the intersections of this surface by planes parallel to that of XY: they are called *sections* of the surface. The section by the plane of XY itself is found by making $z=0$, and is called the *trace* of the surface upon that plane.

If we give a fixed value to y in the equation, we obtain an equation between x and z, which represents the section of the surface by a plane parallel to that of XZ; and if we make $y=0$, we obtain the section by the plane of XZ, or the trace of the surface upon that plane. Similar remarks apply to the results of giving fixed values to x. In the example, $x^2+y^2+z^2=25$, each of the traces is a circle whose centre is at the origin and whose radius is five units in length. The centre of the varying circle, which we regarded in the last Article as describing the surface, is always in the axis of Z; and we may consider the variation in its radius to be regulated by the trace in the plane of XZ, of which a quadrant is drawn in the figure. For this reason, this trace is sometimes called the *director*, while the varying curve is the *generator* of the surface: in the example, the surface generated is that of a sphere.

373. It was shown in Art. 369 that the degree of an equation between x, y and z; that is, of the equation of a surface, cannot be changed by transformation of co-ordinates. Surfaces are therefore classified according to the degrees of their equations. Thus, $2x - y + 3z - 6$ represents a surface of the first degree, and $x^2 - 2xy = 4z - y$, one of the second degree. The section of a surface by any plane may be found, by transforming to a new system of co-ordinates in which the plane of XY is parallel to the given plane, and then giving the proper value to z in the new equation. Now it is evident that the degree of the section cannot exceed that of the surface. Therefore every plane section of a surface of the first degree is a straight line, and every plane section of a surface of the second degree is either a conic or a straight line.

Equations of the Plane.

374. The general equation of the first degree is

$$Ax + By + Cz + D = 0;$$

and, because it is shown in the last Article that every plane intersects a surface of the first degree in a straight line, this is the *general equation of the plane.*

The traces of the plane represented by this equation upon the co-ordinate planes are the straight lines, $Ax + By + D = 0$, $Ax + Cz + D = 0$ and $By + Cz + D = 0$. The intercepts of the plane upon the axes, which are also the intercepts of these lines, are

$$x_0 = -\frac{D}{A}, \qquad y_0 = -\frac{D}{B}, \qquad z_0 = -\frac{D}{C}.$$

If $D = 0$, all the intercepts become zero, and the plane passes through the origin; in fact, whatever the degree of an equation, if there is no absolute term it is satisfied by making $x = 0$, $y = 0$ and $z = 0$, and therefore the surface it represents passes through the origin.

If one of the coefficients A, B or C is zero, so that one of the variables is wanting in the equation, the corresponding intercept becomes infinite. Thus, if $C = 0$, $z_0 = \infty$; in other words, the plane does not cut the axis of Z. Therefore the equation $Ax + By + D = 0$, regarded as the equation of a surface, represents a plane parallel to the axis of Z. This is also the equation of the trace upon the plane of XY, but since z does not enter the equation, it may have any value we please. The trace on the plane of XZ is, in this case, $Ax + D = 0$, which represent a line in that plane parallel to the axis of Z: the trace on the plane of YZ is $By + D = 0$, which represents a line in that plane also parallel to the axis of Z.

In like manner, an equation which wants the term By represents a plane parallel to the axis of Y, and one without the term Ax, a plane parallel to the axis of X.

375. The equation of the plane is readily expressed in terms of its intercepts. Let the given values of the intercepts be a, b and c; then $a = -\frac{D}{A}$, whence $A = -\frac{D}{a}$. Finding in the same manner values for B and C, and substituting in the general equation, we obtain

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

which is the required equation.

The general equation contains four constants, whose ratios determine the values of the intercepts, and therefore fix the position of the plane. These ratios constitute three arbitrary or independent constants, whose values may be so determined as to make the plane fulfil three conditions. Thus, a plane may be made to pass through three given points; as $(2, -1, 5)$, $(3, 2, 1)$ and $(-1, 2, -1)$. For assume the equation in the form $x + by + cz + d = 0$; then we have three equations of condition, from which by elimination we find $b = -3$, $c = -2$, $d = 5$, and the required equation is $x - 3y - 2z + 5 = 0$.

Examples.—Find the equation of the plane passing through $(1, 0, -2)$, $(3, 2, -1)$ and $(5, -1, 2)$; and give the values of its intercepts.

Give the equation of the plane whose intercepts are $3, -1$, and 6.

376. When the axes are rectangular, a convenient form of the

equation of the plane is that in which the constants are the length of the perpendicular from the origin upon the plane and the direction cosines of this perpendicular. Let P be any point of the plane, and O the origin; and join OP. Let p denote the length of a perpendicular to the plane from the origin, and α, β and γ its direction angles. The projection of OP upon the perpendicular is p, and this is equal to the sum of the projections of x, y and z upon the same line. But the projection of x upon this line is $x \cos \alpha$, and those of y and z are $y \cos \beta$ and $z \cos \gamma$. Therefore,

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p.$$

To reduce a given equation to this form, we must divide both members by the square root of the sum of the squares of the coefficients of x, y and z, in order to make the coefficients fulfil the fundamental condition of direction cosines in Art. 361. For example, given $4x - 7y - 4z = 12$, we must divide by 9: the result is $\frac{4}{9}x - \frac{7}{9}y - \frac{4}{9}z = \frac{4}{3}$; therefore, the length of the perpendicular from the origin is $\frac{4}{3}$, and its direction cosines are $\frac{4}{9}$, $-\frac{7}{9}$ and $-\frac{4}{9}$. If the absolute term in the second member is negative, the coefficients will be the negatives of the direction cosines of the perpendicular, which, as shown in Art. 362, belong to the opposite direction, or that of the perpendicular produced.

EXAMPLES.—Reduce to the above form $3x - 5y + 4z + 10 = 0$, and show that the perpendicular makes an angle of 45° with the axis of Y.

Find the perpendicular from the origin on $6x - 2y - 9z + 2 = 0$.

377. The general equation $Ax + By + Cz + D = 0$ is reduced to the form $x \cos \alpha + y \cos \beta + z \cos \gamma = p$ by dividing by $\sqrt{A^2 + B^2 + C^2}$. If the equation contains but two of the variables, one of the direction cosines is zero. Thus, if $C = 0$, $\cos \gamma = 0$ or $\gamma = 90°$, and the perpendicular lies in the plane of XY; the equation in this case takes the form $x \cos \alpha + y \sin \alpha = p$, which represents a plane perpendicular to the plane of XY, or parallel to the axis of Z. If the equation contains but one of the variables, two of the direction cosines vanish. Thus, if $A = 0$ and $B = 0$, $\alpha = 90°$ and $\beta = 90°$, and the perpendicular coincides with the axis of Z; the equation in this case takes the form $\pm z = p$, which represents a plane parallel to the plane of XY.

If from a point on the intersection of two planes perpendiculars to the planes are drawn, and a plane is passed through them, it will cut the given planes in lines perpendicular to the line of intersection. The angle between these lines measures the inclination of the planes, and it is evidently the same as that between the perpendiculars to the planes. Therefore the inclination of two planes is the same as that of the perpendiculars to the planes. Therefore, when the direction cosines of the perpendiculars to two planes are found, the angle between the planes may be found by the formula of Art. 363.

378. If two planes are parallel, their traces upon each of the co-ordinates planes are parallel, and therefore the coefficients of x, y and z in their equations must be proportional. Thus, $x + 2y - 2z + 1 = 0$ and $3x + 6y - 6z = 5$ represent parallel planes.

When the axes are rectangular, the coefficients A, B and C are proportional to the direction cosines of the perpendicular; therefore the condition that two planes shall be perpendicular (from the formula of Art. 364) is

$$AA' + BB' + CC' = 0.$$

Thus, the planes $x + 2y - 2z + 1 = 0$ and $2x + 5y + 6z = 5$ are perpendicular.

We saw in Art. 375 that a plane may be made to fulfil three conditions. Now, to be parallel to a given plane, or to be perpendicular to a given line, is equivalent to two conditions, because it determines the ratios of the three coefficients A, B and C. But to be perpendicular to a given plane is equivalent to a single condition imposed upon the coefficients, and two such conditions will determine the ratios. Thus, if a plane is to be perpendicular to the planes $x + 2y - 2z + 1 = 0$ and $3x - y + 2z = 6$, the conditions give $A + 2B - 2C = 0$ and $3A - B + 2C = 0$. By elimination, we find $B = -4A$ and $C = -\frac{7}{2}A$; therefore the coefficients are as the numbers $2, -8$ and -7. Since the value of one of the coefficients may be assumed at pleasure, the required equation is $2x - 8y - 7z + D = 0$, in which the absolute term may be so determined as to make the plane pass through a given point.

379. The general equation of the plane passing through a given

point, P′, (found by eliminating D from the general equation by an equation of condition) is

$$A(x-x')+B(y-y')+C(z-z')=0.$$

This equation is evidently satisfied by the point P′. It might also have been found by substituting, in the equation of a plane passing through the origin, the values of the co-ordinates of P referred to P′ as a new origin. For, by Art. 365, $x-x'$, $y-y'$ and $z-z'$ are the values of the new co-ordinates; and $AX+BY+CZ=0$ is the equation of a plane passing through the new origin.

In the same manner it may be shown that

$$(x-x')\cos\alpha+(y-y')\cos\beta+(z-z')\cos\gamma=0$$

is the equation of a plane passing through P′ and perpendicular to the line whose direction angles are α, β and γ.

380. If $Ax+By+Cz+D=0$ and $A'x+B'y+C'z+D'=0$ are the equations of two given planes, then

$$Ax+By+Cz+D+k(A'x+B'y+C'z+D')=0$$

is the equation of a plane passing through the straight line in which these planes intersect; for it is evidently satisfied by all the points which satisfy both the given equations, and being of the first degree it represents a plane. The value of k is arbitrary, and may be so determined as to make the plane fulfil another condition; for instance, that of passing through a given point. It may also be so determined as to eliminate one of the variables from the equation, thus making the plane parallel to the corresponding axis, as shown in Art. 374.

EXAMPLES.—Give the general equation of the plane passing through the intersection of $2x+y-3z+1=0$ with $x-2y+z+3=0$, and determine the plane so as to pass through the point $(3, 2-1)$.

Determine the planes passing through the same intersection, and parallel respectively to the axes of X, Y and Z.

Determine the plane passing through the same line, and perpendicular to the plane $3x+2y+2z=0$. (Substitute the coefficients of this equation, and those of the general equation containing k, in the condition of Art. 378.)

Equations of the Straight Line.

381. Since a single equation between x, y and z restricts a point to a certain surface, two such equations taken together restrict a point to the line common to the surfaces; that is, to their line of intersection. Two equations of the first degree, therefore, restrict a point to the intersection of two planes; that is, to a certain straight line.

The position of a straight line is in reality determined by the equations of any two planes passing through it. Therefore, if $Ax + By + Cz + D = 0$ and $A'x + B'y + C'z + D' = 0$ are the equations of two planes passing through a given line, any two of the series of equations which result from giving different values to k in the equation of Art. 380 will determine the same line. The equations of simplest form are those which result from eliminating one of the variables. Thus, given the equations, $2x + y - 3z + 1 = 0$ and $x - 2y + z + 3 = 0$; the simplest equations of planes passing through the line determined by them, are $x - y + 2 = 0$, $x - z + 1 = 0$ and $y - z - 1 = 0$, found by eliminating respectively z, y and x. Thus, there are three equations, of equal simplicity, any *two* of which would serve to determine the line in question and may therefore be considered as its *equations*.

382. Generally, two of the variables in the equations of a line may be expressed as functions of the third. Thus, in the example, the first two of the results of elimination may be written in the forms, $y = x + 2$, $z = x + 1$. (The equation between y and z is not independent of these, but may be derived directly from them.) Therefore x may be considered as an independent variable, for which a value being assumed at pleasure, the corresponding values of y and z are determined by the equations. If, for instance, we assume $x = 1$, the corresponding value of y is found to be 3, and that of z to be 2, therefore the point (1, 3, 2) is a point on the line, as may be verified by showing that it satisfies each of the original equations.

If, however, the line is parallel to one of the co-ordinate planes, one of the co-ordinates is constant; the equations of the line must then express the value of this co-ordinate, and a relation between the values of the other two. This is the case with the line in which $x - y - 2z + 5 = 0$ and $x + 2y + 4z - 4 = 0$ intersect; for if we eliminate z we shall at the same time eliminate y, the result being $3x + 6 = 0$ or $x = -2$. The line is therefore situated in the plane $x = -2$, which is parallel to the plane of YZ. Eliminating x between the equations, we obtain $y + 2z - 3 = 0$; and the equations of the line are $x = -2$ and $y + 2z - 3 = 0$.

Since a straight line is always given by means of the equations of two planes passing through it, the formula of Art. 380 may be used to find the equation of a plane passing through a given straight line and fulfilling another condition.

EXAMPLES.—Determine the plane passing through the line $y = 2x - 3$, $z = 5 - x$ and through the point $(1, -1, 3)$.

Show that, if a line is parallel to the plane of YZ, the traces of all planes passing through it on the plane of YZ will be parallel.

383. *To find the equations of the line passing through two given points.* Let P′ and P″ be the given points, and let P be any point of the line passing through them. Let planes parallel to the plane of YZ be passed through P, P′ and P″. The parts of the axis of X cut off by these planes are the co-ordinates x, x' and x''; therefore the parts intercepted between the first and second of these planes is $x - x'$, and that between the third and second is $x'' - x'$. The corresponding segments of the line are PP′ and P″P′. Now the segments of two lines included between parallel planes are proportional; therefore $x - x' : x'' - x' :: \text{PP}' : \text{P}''\text{P}'$, or $\dfrac{x - x'}{x'' - x'} = \dfrac{\text{PP}'}{\text{P}''\text{P}'}$. In like manner we may prove $\dfrac{y - y'}{y'' - y'} = \dfrac{\text{PP}'}{\text{P}''\text{P}'}$ and $\dfrac{z - z'}{z'' - z'} = \dfrac{\text{PP}'}{\text{P}''\text{P}'}$, Hence

$$\frac{x - x'}{x'' - x'} = \frac{y - y'}{y'' - y'} = \frac{z - z'}{z'' - z'}.$$

This is equivalent to two independent equations; the equality of the first and second members being the equation between x and y, and that of the first and third, the equation between x and z. P′ satisfies these equations because it reduces each member to zero, and P″, because it makes each member unity.

EXAMPLES.—Find the equations of the straight line passing through $(5, 3, -1)$ and $(-2, 1, 4)$, and express y and z as functions of x.

Give the general equation of the plane passing through these points, and determine k so that the plane shall also pass through $(1, 2, 1)$.

384. If we put L, M and N in place of $x''-x'$, $y''-y'$ and $z''-z'$, we have

$$\frac{x-x'}{\mathrm{L}}=\frac{y-y'}{\mathrm{M}}=\frac{z-z'}{\mathrm{N}},$$

in which L, M and N may have any values whatever. These are therefore the general equations of the straight line passing through the given point P′. The direction of the line depends upon the ratios of L, M and N, and not upon their absolute values.

If the point P″ coincide with P′, the value of each of these quantities is zero, and the direction of the line is indeterminate. If P″ and P′ have one common co-ordinate, the line is parallel to one of the co-ordinate planes. Thus, if $x''=x'$ or $\mathrm{L}=0$, the numerator of the first member must be zero, otherwise the fraction takes the infinite form. Therefore in this case, $x-x'=0$, and the equations of the line are

$$x=x' \qquad \text{and} \qquad \frac{y-y'}{\mathrm{M}}=\frac{z-z}{\mathrm{N}},$$

of which the first expresses that the line is in a plane parallel to the plane of YZ; therefore it cannot meet that plane; in other words, it is parallel to it.

If P′ and P″ have two common co-ordinates, the line is parallel to one of the axes. Thus, if $x''=x'$ and $y''=y'$, that is, if $\mathrm{L}=0$ and $\mathrm{M}=0$, we must have

$$x=x' \qquad \text{and} \qquad y=y'.$$

The equations of the line, therefore, express that it is the intersection of two planes parallel respectively to the plane of YZ and to that of XZ; that is, the line is parallel to the axis of Z.

Examples.—Find the equations of the line passing through $(1, 2, 1)$ and $(3, -2, 1)$; through $(1, 2, 1)$ and $(1, 3, 1)$.

385. When L, M and N have finite values, the equations of the line may be reduced to the forms $y=mx+b$ and $z=nx+c$, in

which $m = \frac{M}{L}$ and $n = \frac{N}{L}$. In these equations, b and c determine the point in which the line cuts the plane of YZ, for the point $(0, b, c)$ satisfies the equations.

In finding the intersection of a straight line with a given plane, or the point common to the line and the plane, we have to combine the two equations of the line and the equation of the plane, in order to determine the values of the three co-ordinates. This is, in fact, the same thing as combining the equations of three planes to find their common point, or the values of x, y and z, which satisfy them simultaneously. It is most convenient to express the values of y and z in terms of x, for the equations of the line, and then to substitute them in that of the plane. Thus, given the line $y = 3x + 1$, $z = -x + 2$ and the plane $2x + y + 3z - 5 = 0$, we find by substitution $2x + 2 = 0$ or $x = -1$. The corresponding values of y and z, determined by the equations of the line, are $y = -2$, $z = 3$. Therefore the required point is $(-1, -2, 3)$, as may be verified in the equation of the plane.

When two straight lines are given by their equations, we have in reality four equations of planes. Therefore, to ascertain whether the lines intersect, find the values of x, y and z, which satisfy three of the equations; if they satisfy the fourth equation also, the lines intersect.

Examples.—Determine the intersection of the line passing through $(5, -3, 2)$ and $(1, 1, -6)$ with the plane $x - 3y + z + 10 = 0$; of the same line with the plane of XY.

Does the line joining $(1, 3, 1)$ and the origin cut that joining $(2, 1, -1)$ and $(-1, 2, 4)$? Does either of these lines cut the line $y = 2$, $z = 5 + x$?

386. Substituting $y = mx + b$, $z = nx + c$ in the general equation of the plane, $Ax + By + Cz + D = 0$, we have

$$(A + Bm + Cn)\,x + Bb + Cc + D = 0.$$

If $A + Bm + Cn = 0$, the value of x derived from this equation generally takes the infinite form, which indicates that the line and plane have no common point; that is, that they are parallel. (But if at the same time $Bb + Cc + D = 0$, x takes the indeterminate form, which indicates that the line is situated in the plane.) Substituting for m and n their general values, $\frac{M}{L}$ and $\frac{N}{L}$, we have, for the condition that a line and plane shall be parallel,

$$AL + BM + CN = 0.$$

Thus, the line $\frac{x - x'}{5} = \frac{y - y'}{3} = \frac{z - z'}{-2}$ is parallel to the plane $x + y + 4z = 0$.

387. If in the equations of Art. 383 the axes are rectangular, the denominators $x'' - x'$, $y'' - y'$ and $z'' - z'$ are the projections of $P''P'$ upon the axes. Therefore, we may substitute for these quantities $P''P' \cos \alpha$, $P''P' \cos \beta$ and $P''P' \cos \gamma$. If we then multiply each member of the equation by $P''P'$, the result is

$$\frac{x - x'}{\cos \alpha} = \frac{y - y'}{\cos \beta} = \frac{z - z'}{\cos \gamma}.$$

These are the equations of the line passing through P', and having the direction angles α, β and γ. Each of the three members is a value of PP', the distance between any point of the line and the fixed point P'.

To reduce to this form the equations of a line given in the general form of Art. 384, we must divide the values of L, M and N by $\sqrt{L^2 + M^2 + N^2}$. We therefore find the direction cosines of a line in a manner similar to that in which we found the direction cosines of the perpendicular to a plane in Art. 376.

388. In the general rectangular equations of the straight line and plane, L, M and N are proportional to the direction cosines of the line, and A, B and C to those of the perpendicular or axis of the plane. The condition found in Art. 386 may therefore be regarded as expressing that the line is perpendicular to the axis of the plane. See Art. 378. The condition for perpendicular lines, found in a similar manner, is

$$LL' + MM' + NN' = 0.$$

To be perpendicular to a given line, or to be parallel to a given plane, is equivalent to a single condition imposed upon the ratios of L, M and N; therefore two such conditions may be fulfilled. But, to be parallel to a given line, or to be perpendicular to a given plane, determines the ratios of L, M and N; because these quantities must then be proportional to the given values of L, M and N, or to those of A, B and C.

389. When the axes are rectangular, the distance of a point from a given point, plane or line may readily be found. Since the square of a line is equal to the sum of the squares of its projections upon the axes, the formula for the distance between two points is

$$P''P' = \sqrt{(x''-x')^2 + (y''-y')^2 + (z''-z')^2}.$$

Let p' denote the distance or perpendicular from P′ to the plane $x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0$. Now the projection of OP′ upon the perpendicular to the plane, found as in Art. 376, is $x'\cos\alpha + y'\cos\beta + z'\cos\gamma$. The difference between this projection and p is equal to p'. Hence

$$p' = x'\cos\alpha + y'\cos\beta + z'\cos\gamma - p.$$

Therefore, to find the perpendicular distance from a point to a given plane, reduce its equation to the above form (dividing by the square root of the sum of the squares of the coefficients), and then substitute the co-ordinates of the point in the first member.

Let p'' denote the distance or perpendicular from P″ to the line $\frac{x-x'}{\cos\alpha} = \frac{y-y'}{\cos\beta} = \frac{z-z'}{\cos\gamma}$. Let a plane perpendicular to the given line be passed through P″: it will contain the perpendicular p''. Let p' denote the portion of the given line intercepted between this plane and the known point of the line, P′; that is, the perpendicular from P′ to the plane. Then p'' and p' will be the sides of a right triangle whose hypothenuse is the distance of the points P″ and P′; therefore $p''^2 = P''P'^2 - p'^2$. The equation of the plane is, by the formula of Art. 379, $(x - x'')\cos\alpha + (y - y'')\cos\beta + (z - z'')\cos\gamma = 0$. The values of p' and P″P′ may be found by the formulæ given above. Thus, given the point (4, 2, 1) and the line $\frac{x-3}{2} = \frac{y+1}{2} = z - 2$; the direction co-sines of the line are $\frac{2}{3}$, $\frac{2}{3}$, $\frac{1}{3}$, and the equation of the plane is $\frac{2}{3}(x-4) + \frac{2}{3}(y-2) + \frac{1}{3}(z-1) = 0$. The perpendicular from P′, $(3, -1, 2)$, is $\frac{7}{3}$, and the distance P″P′ is $\sqrt{11}$; hence $p'' = \sqrt{11 - \frac{49}{9}} = \frac{1}{3}\sqrt{50} = \frac{5}{3}\sqrt{2}$.

Surfaces of the Second Degree.

390. The surface generated by the rotation of any conic section about one of its axes is called a *surface of revolution.* The equation of a surface of revolution may be found, when that of the generating curve is known, in the following manner:

Let the surface be referred to rectangular axes, the line about which the rotation takes place being made the axis of Z. The section of the surface by a plane passing through the axis of Z will then be the generating curve. Let r denote the distance of a point of this curve from the axis of Z; then r and z are the rectangular co-ordinates of P, and the equation of the curve is a relation between r and z. Since the axis of Z is an axis of the curve, the equation will not contain the first power of r. Now by Art. 354, $r^2 = x^2 + y^2$; and since we have a relation between r^2 and z, which is true of every point of the surface, if we substitute $x^2 + y^2$ for r^2, the result will be the equation of the surface. The equations of the surfaces thus described are therefore of the second degree.

For example, $r^2 = 2pz$ is the equation of a parabola whose axis coincides with the axis of Z (z taking the place of x, and r that of y, in the ordinary equation $y^2 = 2px$). Therefore $x^2 + y^2 = 2pz$ is the equation of the surface generated, which is called the *paraboloid of revolution.* The trace of this surface upon either the plane of XZ or that of YZ is a parabola having its vertex at the origin. The section by the plane $z = c$ is a circle whose radius is $\sqrt{2pc}$, which is real when c is positive, and imaginary when c is negative.

391. The surface of a right cone is described by a straight line rotating about a fixed line, which it cuts at a constant angle. Taking the origin at the intersection of the generating line with the fixed line or axis of Z, the equation of the line is $r = mz$, in which m is the tangent of the constant angle. Squaring we have $r^2 = m^2z^2$; therefore the equation of the cone is

$$x^2 + y^2 = m^2z^2.$$

The trace upon the plane of YZ is $y^2 = m^2z^2$ or $y = \pm mz$. It is therefore a pair of straight lines making equal angles with the axis of Z. These lines, in fact, constitute a conic of which an axis coincides with the axis of Z. The section by the plane $z = c$ is a circle whose radius is mc, which is always real; but when $c = 0$, the circle reduces to a single point.

If the generating line be parallel to the axis of Z, the surface generated will be that of the right cylinder. The equation of the generating line will then be $r = b$; and that of the cylinder is

$$x^2 + y^2 = b^2.$$

The equation of the surface of the sphere, described by the rotation of the circle $r^2 + z^2 = R^2$, is

$$x^2 + y^2 + z^2 = R^2,$$

in which R denotes the radius of the sphere.

392. When the rectangular equation of a surface contains only the squares of the variables and an absolute term, it can be put in the form

$$\frac{x^2}{\pm A^2} + \frac{y^2}{\pm B^2} + \frac{z^2}{\pm C^2} = 1.$$

The trace of the surface upon either of the co-ordinate planes will then be an ellipse or hyperbola referred to its centre and axes. If the denominators are all positive, the traces are all ellipses. The surface, in this case, is called an *ellipsoid.* This surface encloses a space, and cuts each of the co-ordinates axes. The intercepts A, B and C are called the *semi-axes* of the ellipsoid. If two of the semi-axes are equal the ellipsoid becomes a *spheroid:* it is called a *prolate* or an *oblate* spheroid according as the third semi-axis is greater or less than either of the others. If all three semi-axes are equal, the equation reduces to that of the sphere.

If two of the denominators are positive and the other negative, two of the traces upon the co-ordinate planes are hyperbolas, and the other is an ellipse. The surface, in this case, cuts two of the axes, but not the third. It is a continuous but not a closed surface, and is called an *hyperboloid of one nappe.* If the two positive denominators are equal, the surface is that generated by the rotation of an hyperbola about its conjugate axis.

If one of the denominators is positive and the other two negative, two of the traces are hyperbolas, and the third is an imaginary ellipse; that is, the surface does not intersect the third plane. The surface then consists of two distinct parts, one on each side of this plane; it is therefore called an *hyperboloid of two nappes.* It intersects only one of the axes. If the two negative denominators are equal, the surface is that generated by the rotation of an hyperbola about its transverse axis.

If all three of the denominators are negative, the traces are all imaginary and the surface disappears.

393. The surface represented by an equation of the above form is symmetrical with respect to each of the co-ordinate planes; for, if we assume values of two of the variables, the equation gives equal positive and negative values of the third. The surface is therefore said to be referred to its *centre* and *axes.*

The equation of a surface of the second degree may generally be reduced to this form, by transformation of co-ordinates. In the first place, the axes may be made rectangular; then the three terms containing the products of the variables may be made to disappear by change in the direction of the axes, for, by Art. 367, the formulæ for passing from one rectangular system to another contain three arbitrary constants. This simplification of the equation is found to be always possible. Then, if the result contains the square of each of the variables, the terms of the first degree may be made to disappear by change of origin, using the formulæ of Art. 365. If now the reduced equation contains an absolute term, it represents one of the central surfaces named in the last Article. But if it takes the form

$$Ax^2 + By^2 + Cz^2 = 0,$$

the surface may be regarded as the vanishing case of an ellipsoid or of an hyperboloid, according as the coefficients are all of the same sign, or two only of the same sign. In the first case, the equation is satisfied only by the origin. In the second case, the traces on two of the co-ordinate planes are pairs of straight lines passing through the centre, and the sections parallel to the third plane are similar ellipses. The surface in this case is that of a *cone.* When the coefficients of the same sign are equal, it is a right cone or cone of revolution.

394. When the equation, after being freed from the products of the variables, contains the squares of only two of the variables, the term containing the first power of the other variable cannot be made to disappear. Suppose, for example, that the equation does not contain z^2, then the term containing z cannot be made to vanish. If this term exists in the equation, we may, however, so determine the constant of transformation, z', as to make the absolute term vanish. The equation will then take the form

$$Ax^2 + By^2 + Kz = 0.$$

The traces of this surface on the planes of XZ and YZ are parabolas whose axes coincide with the axis of Z. If A and B are of the same sign, these parabolas are turned in the same direction, and the sections by planes parallel to the plane of XY, on one side of it, are real ellipses. The surface, in this case, is called the *elliptic paraboloid*. When $A = B$, the sections are circles, and we have the paraboloid of revolution.

When A and B are of opposite signs, the axes of the parabolas have opposite directions. The trace on the plane of XY is a pair of straight lines, and the sections parallel to this plane are hyperbolas. The surface is therefore called the *hyperbolic paraboloid.*

The paraboloids are called *non-central* surfaces. They are symmetrical with respect to two of the co-ordinate planes. The intersection of these planes is the axis of the surface: in the equation, this line is the axis of Z.

If the term containing z does not occur in the equation which we have supposed not to contain z^2, the equation has the form

$$Ax^2 + By^2 + L = 0.$$

The traces on the planes of XZ and YZ are now pairs of parallel lines. The trace upon the plane of XY is an ellipse or an hyperbola, according as A and B have the same or opposite signs; and the sections by parallel planes are all equal to the trace. The surface, in this case, is that of an *elliptic cylinder* or of an *hyperbolic cylinder.* The axis of Z is the axis of the cylinder, and any point on that line may be taken as the centre of the surface. These cylinders are therefore said to have a line of centres.

If $L = 0$, the elliptic cylinder reduces to a straight line, and the hyperbolic cylinder to two planes intersecting in the axis of Z.

395. When the rectangular equation contains the square of only one of the variables, the first power of that variable only can be made to disappear. We may, in this case, generally make the absolute term disappear, and the equation will then take the form

$$Ax^2 + Hy + Kz = 0.$$

The traces on the planes of XY and XZ are parabolas. The trace on the plane of YZ is a straight line passing through the origin, and the sections by parallel planes are single straight lines parallel to the trace. The surface represented is the *parabolic cylinder*. This surface is symmetrical with respect to one only of the co-ordinate planes.

If the terms containing the first powers of y and z, as well as those containing their squares, are wanting, the equation has the form

$$Ax^2 + L = 0.$$

This equation represents a pair of real or imaginary parallel planes, according as A and L have the same or opposite signs. These parallel planes constitute a surface of which the centre may be taken anywhere in the plane of YZ. If $L = 0$, the planes are coincident.

The central surfaces consist of the ellipsoid, the two hyperboloids and the cone; the non-central surfaces of the two paraboloids, the three cylinders and pairs of planes; and these are the only varieties of the surface of the second degree.